中国科学院华罗庚数学重点实验室丛书

华罗庚文集

多复变函数论卷 II

华罗庚 / 著

周向宇 / 审校

科学出版社

中国科学院华罗庚数学重点实验室丛书

华罗庚文集

多复变函数论卷 II

华罗庚　　著
周向宇　审校

科学出版社
北京

内 容 简 介

本卷由华罗庚先生的著作《从单位圆谈起》以及一些关于多复变函数论等方面的论文组成.

本书适于科研院所及高等学校数学系师生与数学工作者阅读.

图书在版编目(CIP)数据

华罗庚文集：多复变函数论卷 II/华罗庚著；周向宇审校. —北京：科学出版社，2013

(中国科学院华罗庚数学重点实验室丛书)

ISBN 978-7-03-036011-3

I. 华… II. 华… III. ① 数学-文集 ② 多复变函数论-文集 IV. O1-53

中国版本图书馆 CIP 数据核字(2012)第 267120 号

责任编辑：赵彦超 / 责任校对：钟 洋
责任印制：钱玉芬 / 封面设计：黄华斌

科学出版社 出版
北京东黄城根北街 16 号
邮政编码：100717
http://www.sciencep.com

北京佳信达欣艺术印刷有限公司 印刷
科学出版社发行 各地新华书店经销

*

2013 年 1 月第 一 版 开本：787×1092
2013 年 1 月第一次印刷 印张：28 1/4
字数：552 000

定价：98.00 元

(如有印装质量问题，我社负责调换)

纪念华罗庚先生诞辰100周年

《华罗庚文集》序言

2010 年是著名数学家华罗庚先生诞辰 100 周年. 值此机会, 我们编辑出版《华罗庚文集》, 作为对他的美好纪念.

华罗庚先生是他那个时代的国际领袖数学家之一, 也是中国现代数学的主要奠基人和领导者. 无论是在和平建设时期, 还是在政治动荡甚至是战争年代, 他都抱定了为国家和人民服务的宗旨, 为中国数学的发展倾注了毕生精力, 受到了中国人民的广泛尊敬.

华罗庚先生最初研究数论, 后将研究兴趣拓展至代数和多复变等多个领域, 取得了一系列国际一流的成果, 引领了这些领域的学术发展, 产生了广泛持久的影响. 他从一名自学青年成长为著名数学家, 其传奇经历激励了几代中国数学家投身于数学事业.

华罗庚先生为我们留下了丰富的精神遗产, 包括大量的学术著作和研究论文. 我们认为, 认真研读这些著作和论文, 是深刻把握华罗庚学术思想精髓的最佳途径. 无论对于数学工作者还是青年学生, 其中许多内容都是很有启发和裨益的.

华罗庚先生担任中国科学院数学研究所所长 30 余年, 他言传身教, 培养和影响了一批国际水平的数学家, 他的学术思想和治学精神已经成为数学所文化的核心. 自 2008 年起以中国科学院数学所为基础成立的中国科学院华罗庚数学重点实验室, 旨在继承和弘扬华罗庚先生的学术思想和治学精神, 积极推动中国数学的发展. 为此, 我们选择华罗庚先生的著作和论文作为实验室的首批出版物, 今后还将陆续推出更多优秀的数学出版物.

在出版《华罗庚文集》的过程中, 我们得到了各方面的关心和支持, 包括国家出版基金的资助, 在此我们表示深深的感谢. 同时, 对于有关人员在策划、翻译和审校等方面付出的辛勤劳动, 对于科学出版社所作的大量工作, 我们表示诚挚的谢意.

中国科学院华罗庚数学重点实验室

《华罗庚文集》编委会

2010 年 3 月

目　　录

华罗庚文集 · 多复变函数论卷 II · 上部

华罗庚文集 · 多复变函数论卷Ⅱ · 下部

华罗庚文集·多复变函数论卷Ⅱ·上部

从单位圆谈起

说　　明

这本讲义是根据华罗庚同志 1962 年在中国科学技术大学及中山大学的讲稿由我们整理而成的. 由于我们水平有限, 在整理过程中难免有不妥及错误之处, 望读者指正.

吴兹潜　林　伟　龚　升

1975 年 10 月

第 1 讲　调和函数的几何理论

1.1 旧 事 重 提

在复平面上变形 ①

$$w = \frac{z-a}{1-\overline{a}z}, \quad |a| < 1 \tag{1}$$

及

$$w = e^{i\theta} z. \tag{2}$$

由 (1) 推得

$$1 - |w|^2 = 1 - \frac{(z-a)(\overline{z}-\overline{a})}{(1-\overline{a}z)(1-a\overline{z})} = \frac{(1-|a|^2)(1-|z|^2)}{|1-a\overline{z}|^2}. \tag{3}$$

因此 (1) 把单位圆 $|z| = 1$ 变为单位圆 $|w| = 1$, 单位圆内部变为单位圆内部. 变形 (2) 也有此性质. 并且 (1) 把 $z = a$ 变为 $w = 0$.

微分 (1) 式得

$$dw = \frac{dz}{1-\overline{a}z} + \frac{(z-a)\overline{a}dz}{(1-\overline{a}z)^2} = \frac{1-a\overline{a}}{(1-\overline{a}z)^2}dz. \tag{4}$$

(3)、(4) 相除, 取绝对值的平方得出经过 (1)、(2) 不变的微分型

$$\frac{|dw|^2}{(1-|w|^2)^2} = \frac{|dz|^2}{(1-|z|^2)^2}. \tag{5}$$

与此微分二次型相对应的有不变的微分算子

$$(1-|w|^2)^2 \frac{\partial^2 \Phi}{\partial w \partial \overline{w}} = (1-|z|^2)^2 \frac{\partial^2 \Phi}{\partial z \partial \overline{z}}. \tag{6}$$

这就是 Laplace 算子

$$4\frac{\partial^2 \Phi}{\partial z \partial \overline{z}} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}.$$

(1) 既然把单位圆变为单位圆, 则当 $z = e^{i\tau}(0 \leqslant \tau \leqslant 2\pi)$ 时, $w = e^{i\psi}$, 即

$$e^{i\psi} = \frac{e^{i\tau}-a}{1-\overline{a}e^{i\tau}} = \frac{1-ae^{-i\tau}}{1-\overline{a}e^{i\tau}}e^{i\tau}$$

① 这里 $\overline{a}$ 代表 a 的共轭数.

这代表变形 (1) 在单位圆圆周上所引起的变化. 而 (4) 式变为

$$e^{i\psi}d\psi = \frac{1-a\overline{a}}{(1-\overline{a}e^{i\tau})^2}e^{i\tau}d\tau.$$

两者相除得出

$$d\psi = \frac{1-a\overline{a}}{|1-\overline{a}e^{i\tau}|^2}d\tau. \tag{7}$$

命

$$a = \rho e^{i\theta},\quad \rho < 1$$

及

$$P(\rho,\theta-\tau) = \frac{1-|a|^2}{|1-\overline{a}e^{i\tau}|^2} = \frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2}. \tag{8}$$

这个函数称为 Poisson 核, 因此, Poisson 核是单位圆经 (1) 变为自己所得出的函数行列式. Poisson 核有以下的特点:

(i) 定正性. 当 $\rho < 1$ 时, $P(\rho,\theta-\tau) > 0$.

(ii) $\lim\limits_{\rho\to 1} P(\rho,\theta-\tau) = \begin{cases} 0, & 若\ \theta \neq \tau, \\ \infty, & 若\ \theta = \tau. \end{cases}$

(iii) $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P(\rho,\theta-\tau)d\tau = 1.$

这结果也是显然的, 其理由是, 由 (7) 得

$$\frac{1}{2\pi}\int_0^{2\pi} P(\rho,\theta-\tau)d\tau = \frac{1}{2\pi}\int_0^{2\pi} d\psi = 1.$$

性质 (ii) 与 (iii) 合并称为"δ 函数的性质".

(iv) 当 $\rho < 1$ 时, 它适合于 Laplace 方程 (极坐标形式)

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial^2 u}{\partial\theta^2} = 0. \tag{9}$$

要证明这一点也是十分容易的, 因为

$$\begin{aligned} P(\rho,\theta-\tau) =& 1 + \frac{\rho e^{i(\theta-\tau)}}{1-\rho e^{i(\theta-\tau)}} + \frac{\rho e^{-i(\theta-\tau)}}{1-\rho e^{-i(\theta-\tau)}} \\ =& 1 + 2\sum_{n=1}^{\infty}\rho^n\cos n(\theta-\tau), \end{aligned}$$

而 $\rho^n\cos n(\theta-\tau)$ 显然适合于 (9), 因而 $P(\rho,\theta-\tau)$ 也适合于 (9).

解单位圆的 Dirichlet 问题.

给一个以 2π 为周期的连续函数 $\varphi(\theta)$, 求一函数 $u(\rho e^{i\theta})$ 在圆内适合于 Laplace 方程 ①, 且

$$\lim_{\rho\to 1} u(\rho e^{i\theta}) = \varphi(\theta). \tag{10}$$

这就是有名的 Dirichlet 问题.

我们分以下几个步骤来解决这一问题:

(1) 先证"均值公式": 如果 $u(\rho e^{i\theta})$ 在圆内有二阶连续偏微商, 而且适合 Laplace 方程 (9), 在圆内及圆周上连续, 则

$$\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})d\theta = u(0), \quad 0 \leqslant \rho \leqslant 1. \tag{11}$$

证法是: 由 Laplace 方程知

$$\begin{aligned}\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\cdot\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})d\theta\right) &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2}{\partial\theta^2}u(\rho e^{i\theta})d\theta \\ &= -\frac{1}{2\pi}\left.\frac{\partial}{\partial\theta}u(\rho e^{i\theta})\right|_0^{2\pi} = 0.\end{aligned}$$

求积分得

$$\rho\frac{\partial}{\partial\rho}\cdot\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})d\theta = k.$$

当 $\rho = 0$ 时, 可见 $k = 0$. 再积分, 得

$$\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\theta})d\theta = c.$$

是一与 ρ 无关的常数, 再取 $\rho = 0$, 得 (11) 式.

(2) 依 (1) 换变数, 命

$$v(z) = u(w),$$

则

$$v(e^{i\tau}) = u(e^{i\psi}), \quad v(a) = u(0).$$

(11) 式变为 $(\rho = 1)$

$$\begin{aligned}v(a) =& u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})d\psi \\ =& \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\tau})\frac{1-|a|^2}{|1-\overline{a}e^{i\tau}|^2}d\tau.\end{aligned}$$

① 适合 Laplace 方程的函数称为调和函数.

命 $a = \rho e^{i\theta}$ 及换符号则得 Poisson 公式

$$u(\rho e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\tau})\frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2}d\tau. \tag{12}$$

换言之, 如果 $u(\rho e^{i\theta})$ 是一个调和函数, 则有以上的公式.

(3) 最大 (最小) 值原理. 一个单位圆内的调和函数, 如果不是常数, 则一定在圆周上取最大 (最小) 值.

如果 $u(\rho e^{i\theta})$ 最大, 由 (12) 可知

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\tau})\frac{1-\rho^2}{1-2\rho\cos(\theta-\tau)+\rho^2}d\tau\\ \leqslant & u(\rho e^{i\theta})\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)d\tau}{1-2\rho\cos(\theta-\tau)+\rho^2} = u(\rho e^{i\theta}),\end{aligned}$$

并且仅当 u 是常数时取等号, 不然, 总有一段弧, 其中 $u(e^{i\tau}) < u(\rho e^{i\theta})$, 因而上式取不等号.

同样最小值也在圆周上取.

(4) Dirichlet, 问题解答的唯一性.

如果有两个解 $u(\rho e^{i\theta}), v(\rho e^{i\theta})$ 适合于 (10), 则

$$w(\rho e^{i\theta}) = u(\rho e^{i\theta}) - v(\rho e^{i\theta})$$

也是调和函数, 在圆周上这函数等于 0, 即 $w(e^{i\theta}) = 0$. 由 (3) 可知在闭圆 $|z| \leqslant 1$ 上, $w(\rho e^{i\theta})$ 的最大值 $\leqslant 0$, 最小值 $\geqslant 0$, 因而 $w \equiv 0$. 因而解答是唯一的.

(5) 解答的存在性.

考虑 Poisson 积分

$$u(\rho e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(\rho,\theta-\tau)\varphi(\tau)d\tau. \tag{13}$$

这函数有以下的一些性质: 首先由性质 (iv) 可知 $u(\rho e^{i\theta})$ 在圆内适合 Laplace 方程, 其次由“δ 函数”性质可以证明 (10) 式. 由性质 (iii),

$$\varphi(\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(\rho,\theta-\tau)\varphi(\theta)d\tau.$$

因为 $\varphi(\theta)$ 是连续函数, 给了 ε, 存在 δ 使 $|\theta-\tau| < \delta$ 时,

$$|\varphi(\theta) - \varphi(\tau)| < \varepsilon. \tag{14}$$

把积分

$$u(\rho e^{i\theta}) - \varphi(\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(\rho,\theta-\tau)(\varphi(\tau)-\varphi(\theta))d\tau$$

分为两部分, 由 (14) 可知

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_{|\theta-\tau|<\delta}P(\rho,\theta-\tau)(\varphi(\tau)-\varphi(\theta))d\tau\right|\\ \leqslant&\varepsilon\cdot\frac{1}{2\pi}\int_0^{2\pi}P(\rho,\theta-\tau)d\tau=\varepsilon.\end{aligned}$$

另一方面, 当 $|\theta-\tau|\geqslant\delta$ 时, 可以取 ρ 充分接近于 1 使

$$P(\rho,\theta-\tau)<\varepsilon/2M,$$

这里 M 是 $|\varphi(\tau)|$ 的上界. 于是

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_{|\theta-\tau|\geqslant\delta}P(\rho,\theta-\tau)(\varphi(\tau)-\varphi(\theta))d\tau\right|\\ <&2M\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\varepsilon}{2M}d\tau=\varepsilon.\end{aligned}$$

合并之, 得出当 ρ 充分接近于 1 时,

$$|u(\rho e^{i\theta})-\varphi(\theta)|<2\varepsilon,$$

即

$$\lim_{\rho\to1}u(\rho e^{i\theta})=\varphi(\theta).$$

因而公式 (13) 解决了单位圆的 Dirichlet 问题的存在性部分.

1.2 实数形式

为了看出推广的可能性, 先看 1.1 节的结果的实数形式, 先看变形 (1.1) 的实数形式:

$$w=\frac{z-a}{1-\overline{a}z}=\frac{(z-a)(1-a\overline{z})}{(1-\overline{a}z)(1-a\overline{z})}=\frac{z-a-az\overline{z}+a^2\overline{z}}{1-\overline{a}z-a\overline{z}+a\overline{a}z\overline{z}}.$$

把复数 v 写成为 $\xi+i\eta$, 而以 v^* 代表矢量 (ξ,η), 显然有

$$a\overline{b}+\overline{a}b=2a^*b^{*\prime}.$$

又由于

$$a^2\overline{z}=(b^2-c^2+2bci)(x-iy)\quad(a=b+ic),$$

所以

$$(a^2\overline{z})^*=[(b^2-c^2)x+2bcy,2bcx-(b^2-c^2)y]$$

$$
\begin{aligned}
&=(x,y)\begin{pmatrix} b^2-c^2 & 2bc \\ 2bc & -b^2+c^2 \end{pmatrix} \\
&=(x,y)[2(b,c)'(b,c)-(b,c)(b,c)'I] \\
&=z^*(2a^{*\prime}a^*-a^*a^{*\prime}I),
\end{aligned}
$$

这里依照矩阵相乘的法则办事, 因此得

$$
w^*=\frac{z^*-a^*-z^*z^{*\prime}a^*+z^*(2a^{*\prime}a^*-a^*a^{*\prime}I)}{1-2a^*z^{*\prime}+a^*a^{*\prime}z^*z^{*\prime}}.
$$

1.3　单位球的几何学

以上的实形式建议以下的可能推广

命 $x=(x_1,\cdots,x_n)$ 代表一 n 维矢量, 而

$$
xx'<1 \tag{1}
$$

代表一单位球, 以上建议

$$
y=\frac{x-a-xx'a+x(2a'a-aa'I)}{1-2ax'+aa'xx'},\quad aa'<1 \tag{2}
$$

可能是一个变形把单位球一对一地变为其自己, 而且把 $x=a$ 变为 $y=0$.

先把 y 写成为

$$
y=\frac{(1-aa')(x-a)-a(x-a)(x-a)'}{1-2ax'+aa'xx'},\quad aa'<1. \tag{3}
$$

作内积

$$
\begin{aligned}
yy'=&\frac{(1-aa')^2(x-a)(x-a)'}{(1-2ax'+aa'xx')^2} \\
&-\frac{2(1-aa')(x-a)(x-a')a(x-a)'}{(1-2ax'+aa'xx')^2}+\frac{aa'[(x-a)(x-a)']^2}{(1-2ax'+aa'xx')^2} \\
=&\frac{(x-a)(x-a)'}{(1-2ax'+aa'xx')^2}[(1-aa')^2 \\
&-2(1-aa')(x-a)a'+aa'(x-a)(x-a)'] \\
=&\frac{(x-a)(x-a)'}{1-2ax'+aa'xx'}.
\end{aligned} \tag{4}
$$

由 (3) 与 (4) 可知

$$
y+yy'a=\frac{(1-aa')(x-a)}{1-2ax'+aa'xx'}, \tag{5}
$$

再作内积

$$(y+yy'a)(y+yy'a)'=\frac{(1-aa')^2(x-a)(x-a)'}{(1-2ax'+aa'xx')^2}.$$

由 (4) 得

$$yy'(1+2ay'+aa'yy')=\frac{(1-aa')^2yy'}{1-2ax'+aa'xx'}.$$

如果 $yy'=0$, 则 $y=0$, 由 (4) 得 $x=a$. 如果 $yy'\neq 0$, 则得等式

$$1+2ay'+aa'yy'=\frac{(1-aa')^2}{1-2ax'+aa'xx'} \tag{6}$$

(这对 $y=0,x=a$ 也对). 代入 (5) 式得

$$x=a+\frac{(y+yy'a)(1-aa')}{1+2ay'+aa'yy'},$$

即

$$x=\frac{y+a+ayy'+y(2a'a-aa'I)}{1+2ay'+aa'yy'}. \tag{7}$$

这与 (2) 的形式完全相同, 只不过把 a 换成 $-a$ 而已. 因此 (2) 的确是一个一对一的变形 (对整个空间都如此, 除去分母为 0 的情况, 不难证明, 例外仅有 $y=-a/(aa')$ 一点而已).

再由 (4) 可知

$$\begin{aligned}1-yy'=&\frac{1-2ax'+aa'xx'-(x-a)(x-a)'}{1-2ax'+aa'xx'}\\=&\frac{(1-aa')(1-xx')}{1-2ax'+aa'xx'}.\end{aligned} \tag{8}$$

由 Schwarz 不等式可知, 分母

$$1-2ax'-aa'xx'=(1-ax')^2+aa'xx'-(ax')^2>0,$$

这又证明了 (2) 把单位球变为其自己.

除形式 (2) 的变形以外, 变形

$$y=x\Gamma,\quad \Gamma\Gamma'=I \tag{9}$$

也显然把单位球变为其自己.

(2) 与 (9) 所演出的群就是我们所要讨论的群. 我们现在是研究在此群下, 单位球内点所成的空间的几何学.

这个空间称为双曲空间, 由 (2) 和 (9) 所演出的群称为非欧运动群.

在此群下, 球内任一点可以变为原点, 而且任意相互正交的 n 个方向可以变为 n 个坐标轴的正向.

1.4 微分度量

求变形

$$y=\frac{(1-aa')(x-a)-(x-a)(x-a)'a}{1-2ax'+aa'xx'}$$

的微分，有

$$\begin{aligned}&(1-2ax'+aa'xx')^2dy\\=&(1-2ax'+aa'xx')\times[(1-aa')dx-2dx(x-a)'a]\\&-[-2dxa'+2aa'dxx'][(1-aa')(x-a)-(x-a)(x-a)'a]\\=&(1-aa')dx\times\{(1-2ax'+aa'xx')I-2(1-2ax')x'a\\&+2a'x-2xx'a'a-2aa'x'x\}\\=&(1-aa')dx\{(1-2ax'+aa'xx')I-2(1-ax')\\&\times(x'a-a'x)+2(x'a-a'x)^2\}.\end{aligned}$$

命

$$P=(1-2ax'+aa'xx')I-2(1-ax')(x'a-a'x)+2(x'a-a'x)^2$$

及

$$M=x'a-a'x,\quad \lambda=1-2ax'+aa'xx',\tag{1}$$

则

$$P=\lambda I-2(1-ax')M+2M^2\tag{2}$$

及

$$dy=\frac{1-aa'}{(1-2ax'+aa'xx')^2}dxP.\tag{3}$$

易证：

$$\begin{aligned}xM^2&=[(ax')^2-aa'xx']x,\\aM^2&=[(ax')^2-aa'xx']a,\\M^3&=[(ax')^2-aa'xx']M.\end{aligned}\tag{4}$$

因此得出

$$\begin{aligned}PP'=&(\lambda I-2(1-ax')M+2M^2)\\&\times(\lambda I+2(1-ax')M+2M^2)\\=&(\lambda I+2M^2)^2-4(1-ax')^2M^2\end{aligned}$$

$$
\begin{aligned}
&=\lambda^2 I + 4(\lambda - (1-ax')^2)M^2 + 4M^4\\
&=\lambda^2 I + 4M\{[aa'xx' - (ax')^2]M + M^3\}\\
&=\lambda^2 I.
\end{aligned}
\tag{5}
$$

因此

$$
\begin{aligned}
dydy' &= \frac{(1-aa')^2}{(1-2ax'+aa'xx')^4} dxPP'dx'\\
&= \frac{(1-aa')^2}{(1-2ax'+aa'xx')^2} dxdx'.
\end{aligned}
\tag{6}
$$

与 (1.3.8) 联立, 立刻推得

$$
\frac{dydy'}{(1-yy')^2} = \frac{dxdx'}{(1-xx')^2}. \tag{7}
$$

这关系也是经过 (1.3.9) 而不变的, 因此 (7) 是一个不变的微分二次型.

1.5 微 分 算 子

今往证明偏微分方程

$$
(1-yy')^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial y_i^2} + 2(n-2)(1-yy') \sum_{i=1}^{n} y_i \frac{\partial u}{\partial y_i} = 0. \tag{1}
$$

经变形 (1.3.2) 而不变, (1) 可以改写为

$$
(1-yy')^n \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left[(1-yy')^{2-n} \frac{\partial u}{\partial y_i} \right] = 0,
$$

即待证

$$
\begin{aligned}
&(1-yy')^n \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left[(1-yy')^{2-n} \frac{\partial u}{\partial y_i} \right]\\
=&(1-xx')^n \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[(1-xx')^{2-n} \frac{\partial u}{\partial x_i} \right]
\end{aligned}
\tag{2}
$$

在证明 (2) 成立前, 先证明以下的一些结果.

引理 1 命 $\mu = 1 + 2ay' + aa'yy'$, 则

$$
\sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left(\frac{1}{\mu^{n-2}} \frac{\partial x_k}{\partial y_i} \right) = 0. \tag{3}
$$

证 由 (1.3.7) 已知

$$x_k = a_k + \frac{(1-aa')(y_k + yy'a_k)}{1+2ay'+aa'yy'},$$

即待证

$$\sum_{i=1}^{n} \frac{\partial}{\partial y_i} \frac{1}{\mu^{n-2}} \frac{\partial}{\partial y_i} \frac{y_k + yy'a_k}{\mu} = 0. \tag{4}$$

左边等于

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\partial}{\partial y_i} \frac{1}{\mu^n} [(\delta_{ik} + 2y_i a_k)(1+2ay'+aa'yy') \\
&- 2(y_k + yy'a_k)(a_i + aa'y_i)] \\
=&\frac{1}{\mu^{n+1}} \sum_{i=1}^{n} \{[2a_k(1+2ay'+aa'yy') + 2(\delta_{ik} + 2y_i a_k) \\
&\times (a_i + aa'y_i) - 2(\delta_{ik} + 2y_i a_k)(a_i + aa'y_i) \\
&- 2(y_k + yy'a_k)aa'](1+2ay'+aa'yy') \\
&- 2n[(\delta_{ik} + 2y_i a_k)(1+2ay'+aa'yy') \\
&- 2(y_k + yy'a_k)(a_i + aa'y_i)](a_i + aa'y_i)\} \\
=&\frac{1}{\mu^{n+1}} \sum_{i=1}^{n} \{[2a_k(1+2ay') - 2y_k aa'](1+2ay' \\
&+ aa'yy') - 2n(\delta_{ik} + 2y_i a_k)(a_i + aa'y_i) \\
&\times (1+2ay'+aa'yy') + 4n(y_k + yy'a_k) \\
&\times (a_i + aa'y_i)(a_i + aa'y_i)\} \\
=&\frac{1}{\mu^n} \{n[2a_k(1+2ay') - 2y_k aa'] - 2n(a_k + aa'y_k \\
&+ 2ay'a_k + 2aa'yy'a_k) + 4n(y_k + yy'a_k)aa'\} = 0.
\end{aligned}$$

引理 2

$$\sum_{i=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial x_k}{\partial y_i} = \frac{\lambda^2}{(1-aa')^2} \delta_{jk}.$$

这是极易从

$$dydy' = \frac{(1-aa')^2}{\lambda^2} dxdx'$$

推得的

现在往证 (2) 式.

由 (1.3.8) 立刻推出

$$\left(\frac{(1-aa')(1-xx')}{\lambda(x)}\right)^n \sum_{i=1}^n\sum_{j=1}^n \frac{\partial}{\partial x_j}\left[\left(\frac{(1-aa')(1-xx')}{\lambda(x)}\right)^{2-n}\sum_{k=1}^n \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial y_i}\right]\frac{\partial x_j}{\partial y_i}$$
$$=(1-aa')^2\left(\frac{1-xx'}{\lambda(x)}\right)^n \sum_{i,j,k}\frac{\partial}{\partial x_j}\left[(1-xx')^{2-n}\frac{\partial u}{\partial x_k}\right]\lambda^{n-2}\frac{\partial x_k}{\partial y_i}\frac{\partial x_j}{\partial y_i}$$
$$+(1-aa')^2\left(\frac{1-xx'}{\lambda(x)}\right)^n \sum_{i,j,k}(1-xx')^{2-n}\frac{\partial u}{\partial x_k}$$
$$\times\frac{\partial}{\partial x_j}\left(\lambda^{n-2}\frac{\partial x_k}{\partial y_i}\right)\frac{\partial x_j}{\partial y_i}=s_1+s_2.$$

由引理 2, s_1 就是 (2) 式的右边, 因此待证 $s_2=0$. 也就是要证明

$$\sum_{i,j,k}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_j}\left(\lambda^{n-2}\frac{\partial x_k}{\partial y_i}\right)\frac{\partial x_j}{\partial y_i}=0.$$

这等式显然易由下面的等式推出:

$$\sum_{i,j}\frac{\partial}{\partial x_j}\left(\lambda^{n-2}\frac{\partial x_k}{\partial y_i}\right)\frac{\partial x_j}{\partial y_i}=0,$$

即

$$\sum_{i=1}^n\frac{\partial}{\partial y_i}\left(\lambda^{n-2}\frac{\partial x_k}{\partial y_i}\right)=0.$$

由 (1.3.6) 得 $\lambda\mu=(1-aa')^2$, 以上的等式显然可由引理 1 推出.

1.6 球 坐 标

命

$$x=\rho u,\quad uu'=1,$$

则 $du\cdot u'=0$. 因此

$$\begin{aligned}dxdx'=&(d\rho u+\rho du)(d\rho u+\rho du)'\\=&d\rho^2+\rho^2dudu'.\end{aligned}$$

所以得到

$$\frac{dxdx'}{(1-xx')^2}=\frac{d\rho^2+\rho^2dudu'}{(1-\rho^2)^2}.\tag{1}$$

引进球坐标

$$u = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots,$$
$$\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}),$$
$$0 \leqslant \theta_1, \theta_2, \cdots, \theta_{n-2} \leqslant \pi, 0 \leqslant \theta_{n-1} \leqslant 2\pi.$$

单位圆的圆周可以表成为 $(\cos\theta, \sin\theta), 0 \leqslant \theta \leqslant 2\pi$, 但并不能说单位圆的圆周就等价于区间 $[0, 2\pi]$, 在区间 $[0, 2\pi]$ 上的连续函数并不一定是单位圆周上的连续函数. 其原因是 $\theta = 0$ 与 $\theta = 2\pi$ 实际上代表同一点, 因此在说到单位圆周上的连续函数 $f(\theta)$ 时, 必须理解到它是一个以 2π 为周期的函数.

在球面上情况更为复杂：因为 $(\cos\theta_1, \sin\theta_1\cos_2, \cdots, \sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1})$ 并不是 θ_1 的以 π 为周期的函数, 在球面上怎样定义一个函数

$$f(\theta_1, \cdots, \theta_{n-1}), \quad 0 \leqslant \theta_1, \cdots, \theta_{n-2} \leqslant \pi, 0 \leqslant \theta_{n-1} \leqslant 2\pi$$

的连续性, 主要看区间端点的情况. 先看 $\theta_1 = 0$ 所代表的点, 不管 $\theta_2, \cdots, \theta_{n-1}$ 如何, 当 $\theta_1 = 0$ 时, 则 $u = e_1 = (1, 0, \cdots, 0)$, 因此在 $u = e_1$ 时球面上的连续函数 $f(\theta_1, \cdots, \theta_{n-1})$ 必须使

$$\lim_{\theta_1\to+0} f(\theta_1, \cdots, \theta_{n-1})$$

存在, 而且与 $\theta_2, \cdots, \theta_{n-1}$ 无关. 同理

$$\lim_{\theta_1\to\pi-0} f(\theta_1, \cdots, \theta_{n-1})$$

是 $u = -e_1$ 时的函数值, 与 $\theta_2, \cdots, \theta_{n-1}$ 无关. 同理

$$\lim_{\theta_2\to+0} f(\theta_1, \theta_2, \cdots, \theta_{n-1})$$

与 $\theta_3, \cdots, \theta_{n-1}$ 无关, 等等. 最后

$$\lim_{\theta_{n-1}\to+0} f(\theta_1, \cdots, \theta_{n-1}) = \lim_{\theta_{n-1}\to 2\pi-0} f(\theta_1, \cdots, \theta_{n-1}).$$

适合这些条件的连续函数才是球面上的连续函数.

微分矢量 u, 极易推出

$$dudu' = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1\sin^2\theta_2 d\theta_3^2 + \cdots$$
$$+ \sin^2\theta_1\cdots\sin^2\theta_{n-2}d\theta_{n-1}^2. \tag{2}$$

单位球表面积的元素 $\dot{u}$ 是这微分二次型的行列式的平方根, 即

$$\dot{u} = \sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}d\theta_1 d\theta_2\cdots d\theta_{n-1},$$

不难算出总表面积等于 $\omega_{n-1}=\dfrac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$.

易知 Laplace 算子 (也不难直接算出)

$$\Delta^2=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2} \tag{3}$$

的极坐标形式是

$$\Delta^2=\frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\partial_u^2, \tag{4}$$

这里

$$\begin{aligned}\partial_u^2=&\frac{\partial^2}{\partial\theta_1^2}+\frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}+\cdots+\frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{n-2}}\frac{\partial^2}{\partial\theta_{n-1}^2}\\&+(n-2)\cot\theta_1\frac{\partial}{\partial\theta_1}+(n-3)\frac{\cot\theta_2}{\sin^2\theta_1}\frac{\partial}{\partial\theta_2}\\&+(n-4)\frac{\cot\theta_3}{\sin^2\theta_1\sin^2\theta_2}\frac{\partial}{\partial\theta_3}+\cdots+\frac{\cot\theta_{n-2}}{\sin^2\theta_1\cdots\sin^2\theta_{n-3}}\frac{\partial}{\partial\theta_{n-2}}.\end{aligned} \tag{5}$$

现在考虑微分算子

$$(1-xx')^2\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}+2(n-2)(1-xx')\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$$

的极坐标形式.

由

$$\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}=\rho\frac{\partial}{\partial\rho}$$

及 (4) 得

$$\begin{aligned}&(1-xx')^2\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}+2(n-2)(1-xx')\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\\=&(1-\rho^2)^2\left[\frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\partial_u^2\right]+2(n-2)(1-\rho^2)\rho\frac{\partial}{\partial\rho}\\=&(1-\rho^2)^2\frac{\partial^2}{\partial\rho^2}+\frac{1-\rho^2}{\rho}[(n-1)+(n-3)\rho^2]\frac{\partial}{\partial\rho}+\frac{(1-\rho^2)^2}{\rho^2}\partial_u^2\\=&\frac{(1-\rho^2)^n}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\frac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\frac{\partial}{\partial\rho}\right)+\frac{(1-\rho^2)^2}{\rho^2}\partial_u^2.\end{aligned} \tag{6}$$

我们考虑 (6) 作用在与 $\theta_2,\theta_3,\cdots,\theta_{n-1}$ 无关的函数 $\Phi(\rho\cos\theta_1,\rho\sin\theta_1)$ 的情况, 得出以下的偏微分方程:

$$\frac{(1-\rho^2)^n}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\frac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\frac{\partial}{\partial\rho}\right)\Phi+\frac{(1-\rho^2)^2}{\rho^2}\left(\frac{\partial^2}{\partial\theta_1^2}+(n-2)\cot\theta_1\frac{\partial}{\partial\theta_1}\right)\Phi=0.$$

命 $\xi=\cos\theta_1$, 则得

$$\frac{(1-\rho^2)^n}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\frac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\frac{\partial}{\partial\rho}\right)\Phi + \frac{(1-\rho^2)^2}{\rho^2}\left[(1-\xi^2)\frac{\partial^2}{\partial\xi^2}-(n-1)\xi\frac{\partial}{\partial\xi}\right]\Phi=0. \tag{7}$$

这也可以写成为

$$\left[\frac{(1-\rho^2)^{n-2}}{\rho^{n-3}}\frac{\partial}{\partial\rho}\left(\frac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\frac{\partial}{\partial\rho}\right) + (1-\xi^2)^{-\frac{n-3}{2}}\frac{\partial}{\partial\xi}\left((1-\xi^2)^{\frac{n-1}{2}}\frac{\partial}{\partial\xi}\right)\right]\Phi=0. \tag{8}$$

这建议在长方形

$$0\leqslant\rho\leqslant 1,\quad -1\leqslant\xi\leqslant 1$$

内研究拟保角变换

$$\begin{cases} u=u(\rho,\xi),\\ v=v(\rho,\xi).\end{cases}$$

这一对函数 u,v 适合于微分方程组

$$\begin{cases} \dfrac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\dfrac{\partial u}{\partial\rho}=(1-\xi^2)^{-\frac{1}{2}(n-3)}\dfrac{\partial v}{\partial\xi},\\ \dfrac{(1-\rho^2)^{n-2}}{\rho^{n-3}}\dfrac{\partial v}{\partial\rho}=-(1-\xi^2)^{\frac{1}{2}(n-1)}\dfrac{\partial u}{\partial\xi}.\end{cases} \tag{9}$$

由 (9) 利用 $\dfrac{\partial^2 v}{\partial\xi\partial\rho}=\dfrac{\partial^2 v}{\partial\rho\partial\xi}$ 消去 v, 即得 u 适合于微分方程 (8). 利用 $\dfrac{\partial^2 u}{\partial\xi\partial\rho}=\dfrac{\partial^2 u}{\partial\rho\partial\xi}$ 消去 u, 得出 v 所适合的微分方程是

$$\left[\frac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\frac{\partial}{\partial\rho}\left(\frac{(1-\rho^2)^{n-2}}{\rho^{n-3}}\frac{\partial}{\partial\rho}\right) + (1-\xi^2)^{\frac{1}{2}(n-1)}\frac{\partial}{\partial\xi}\left((1-\xi^2)^{-\frac{1}{2}(n-3)}\frac{\partial}{\partial\xi}\right)\right]v=0.$$

这两个微分方程的二次项相等, 一次项两者之和等于

$$\frac{\partial v}{\partial\rho}\cdot\frac{\partial}{\partial\rho}\left(\frac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\cdot\frac{(1-\rho^2)^{n-2}}{\rho^{n-3}}\right)+\frac{\partial v}{\partial\xi}\cdot\frac{\partial}{\partial\xi}(1-\xi^2)^{\frac{1}{2}(n-1)-\frac{1}{2}(n-3)}$$
$$=2\rho\frac{\partial v}{\partial\rho}-2\xi\frac{\partial v}{\partial\xi}.$$

(9) 式可能是离普通保角变换最相近的拟保角变换. 换言之, 深入研究这一特殊情况可能作为研究一般拟保角变换的参考.

1.7 Poisson 公式

由

$$dydy' = \left(\frac{1-aa'}{1-2ax'+aa'xx'}\right)^2 dxdx'$$

可知, 在球面 $x = u, y = v, uu' = vv' = 1$ 上也有

$$dvdv' = \left(\frac{1-aa'}{1-2au'+aa'uu'}\right)^2 dudu'.$$

球面积有以下的关系式

$$\dot{v} = \left(\frac{1-aa'}{1-2au'+aa'}\right)^{n-1} \dot{u} \tag{1}$$

(因为 du 是 $(n-1)$ 维矢量).

这建议有以下的 Poisson 公式

$$\Phi(x) = \frac{1}{\omega_{n-1}} \int \cdots \int_{uu'=1} \left(\frac{1-xx'}{1-xu'+xx'}\right)^{n-1} \Phi(u)\dot{u} \tag{2}$$

的可能性, 这建议说, 如果在单位球面 $uu' = 1$ 上给了一个函数 $\Phi(u)$, 我们由 (2) 所定义的函数既在球上适合于 (2), 又在球内适合偏微分方程 (1.5.1). 在详细论述这个问题之前, 先研究 Poisson 核

$$P(x,u) = \left(\frac{1-xx'}{1-2xu'+xx'}\right)^{n-1} \tag{3}$$

的性质, 命 $x = \rho v$, 则

$$P(x,u) = \left(\frac{1-\rho^2}{1-2\rho\cos\langle u,v\rangle+\rho^2}\right)^{n-1}$$

这里 $\langle u,v\rangle$ 表示两个单位矢量 u, v 的夹角.

(1) 当 $0 \leqslant \rho < 1$ 时, $P(x,u) > 0$. 这是显然的, 因为

$$1-2\rho\cos\langle u,v\rangle+\rho^2 \geqslant 1-2\rho+\rho^2 = (1-\rho)^2.$$

(2) 我们有

$$\lim_{\rho\to 1} P(x,u) = \begin{cases} 0, & u \neq v, \\ \infty, & u = v. \end{cases}$$

具体些, 命 $\langle u,v\rangle=\alpha$, 则当 $|\alpha|>\delta$ 时, 给任一 ε 可以找到 ρ_0, 使 $1>\rho>\rho_0$ 时,

$$P(x,u)\leqslant\left(\frac{1-\rho^2}{1-2\rho\cos\delta+\rho^2}\right)^{n-1}<\varepsilon.$$

(3) $\dfrac{1}{\omega_{n-1}}\underset{u}{\int\cdots\int}P(x,u)\dot{u}=1.$

这可由一目了然的公式

$$\frac{1}{\omega_{n-1}}\underset{v}{\int\cdots\int}\dot{v}=1$$

经 (1) 而变得.

(4) 当 $\rho<1$ 时, $P(x,u)$ 是适合于方程 (1.5.1) 的.

微分 $P(x,u)$ 得

$$\frac{\partial P(x,u)}{\partial x_i}=2(n-1)\frac{(1-xx')^{n-2}}{(1-2ux'+xx')^n}\times[-2(1-ux')x_i+(1-xx')u_i].$$

于是

$$\begin{aligned}&\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[(1-xx')^{2-n}\frac{\partial P(x,u)}{\partial x_i}\right]\\
=&2(n-1)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\frac{-2(1-ux')x_i+(1-xx')u_i}{(1-2ux'+xx')^n}\right]\\
=&2(n-1)\sum_{i=1}^{n}\left\{\frac{-2(1-ux')+2u_ix_i-2x_iu_i}{(1-2ux'+xx')^n}\right.\\
&\left.-\frac{n[-2(1-ux')x_i+(1-xx')u_i][-2u_i+2x_i]}{(1-2ux'+xx')^{n+1}}\right\}\\
=&2(n-1)\left\{\frac{-2n(1-ux')}{(1-2ux'+xx')^n}\right.\\
&\left.-\frac{2n[2(1-ux')ux'-2(1-ux')xx'-(1-xx')uu'+(1-xx')ux']}{(1-2ux'+xx')^{n+1}}\right\}.\end{aligned}$$

由于 $uu'=1$, 故上式等于 0, 即 $P(x,u)$ 适合方程 (1.5.1).

定理 1　假定 $\varPhi(u)$ 是一在 $uu'=1$ 上定义的连续函数, Poisson 公式

$$\varPhi(x)=\frac{1}{\omega_{n-1}}\underset{uu'=1}{\int\cdots\int}\left(\frac{1-xx'}{1-2xu'+xx'}\right)^{n-1}\varPhi(u)\dot{u}$$

定义一在单位球内适合于方程 (1.5.1) 的函数, 并且

$$\lim_{r\to1}\varPhi(rv)=\varPhi(v).$$

证 当 $r<1$ 时, 由于 $P(x,u)$ 适合于方程 (1.5.1), 因此用积分号下求微分法可知 $\Phi(x)$ 也适合于方程 (1.5.1).

今往证: 当 $r\to 1$ 时,

$$\Phi(rv)-\Phi(v)=\frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}\left(\frac{1-xx'}{1-2xu'+xx'}\right)^{n-1}\times(\Phi(u)-\Phi(v))\dot{u}$$

趋于 0.

命 $\cos\alpha=vu'$, 把积分分成为

$$\Phi(rv)-\Phi(v)=\frac{1}{\omega_{n-1}}\left(\int\cdots\int_{|\alpha|<\delta}+\int\cdots\int_{|\alpha|\geqslant\delta}\right)=s_1+s_2.$$

取 δ 充分小, 使

$$|\Phi(u)-\Phi(v)|<\varepsilon,$$

则

$$s_1=O\left(\varepsilon\int\cdots\int_{uu'=1}\left(\frac{1-xx'}{1-2xu'+xx'}\right)^{n-1}\dot{u}\right)=O(\varepsilon).$$

对已定的 δ 可取 r 充分接近于 1, 使

$$\left|\frac{1-xx'}{1-2xu'+xx'}\right|^{n-1}\leqslant\varepsilon.$$

因此

$$s_2=O(\varepsilon).$$

即得所证.

1.8 建议了些什么?

以上所讲的至少有三种启发:

1° 是否还有其他的可递群把单位球变为单位球?

2° 从“δ”函数出发 (即把 $P(x,u)$ 的性质抽象出来) 而研究偏微分方程的 Dirichlet 问题.

3° 从边界上的调和分析出发.

我们现在先回答 1°, 2°. 关于问题 3° 留在第二讲中回答.

1° 变形

$$y=\frac{\sqrt{1-aa'}(x-a)(1+\lambda a'a)}{1-ax'},\tag{1}$$

这里 $aa' < 1$, 而

$$\lambda = \frac{1-\sqrt{1-aa'}}{aa'\sqrt{1-aa'}}, \quad 1+\lambda aa' = \frac{1}{\sqrt{1-aa'}}.$$

先证变形 (1) 把单位球变为单位球. 由于

$$\begin{aligned} yy' =& \frac{1-aa'}{(1-ax')^2}(x-a)(I+\lambda a'a)^2(x-a)' \\ =& \frac{1-aa'}{(1-ax')^2}(x-a)\left(I+\frac{1}{1-aa'}a'a\right)(x-a)' \\ =& \frac{(1-aa')(x-a)(x-a)'+[(x-a)a']^2}{(1-ax')^2} \end{aligned}$$

因此

$$\begin{aligned} 1-yy' =& \frac{(1-aa')[(1-2ax'+aa')-(x-a)(x-a)']}{(1-ax')^2} \\ =& \frac{(1-aa')(1-xx')}{(1-ax')^2}. \end{aligned} \tag{2}$$

不难证明

$$\frac{dy(I-y'y)^{-1}dy'}{1-yy'} = \frac{dx(I-x'x)^{-1}dx'}{1-xx'}, \tag{3}$$

即

$$dy(I-y'y)^{-1}dy' = \frac{1-aa'}{(1-ax')^2}dx(I-x'x)^{-1}dx'.$$

在单位球上 $x=u, y=v$, 由于 $duu'=0$, 所以得

$$dvdv' = \frac{1-aa'}{(1-au')^2}dudu'.$$

因此得出 Poisson 核

$$P(x,u) = \frac{(1-xx')^{\frac{1}{2}(n-1)}}{(1-ux')^{n-1}}. \tag{4}$$

$P(x,u)$ 所适合的微分方程是

$$\sum_{i=1}^{n}\frac{\partial^2\Phi}{\partial x_i^2} - \sum_{i,j=1}^{n}x_ix_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} - 2\sum_{i=1}^{n}x_i\frac{\partial\Phi}{\partial x_i} = c. \tag{5}$$

因而证明：Poisson 公式

$$\Phi(x) = \frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}\frac{(1-xx')^{\frac{1}{2}(n-1)}}{(1-ux')^{n-1}}\varphi(u)\dot{u} \tag{6}$$

给出偏微分方程 (5) 的 Dirichlet 问题的解 (证明唯一性后便完全解决了 Dirichlet 问题).

2° 不从群出发, 而仅从 "Ponisson 核" 出发, 也有可能性.

命 $\mathscr{D}$ 是一域, $\mathscr{L}$ 是它的边界, 如果我们可以找到一个函数

$$P(x,u),$$

x 在域 $\mathscr{D}$ 里变, u 在边界 $\mathscr{L}$ 上变, 而且适合以下的一些性质:

(i) $P(x,u)>0$;

(ii) $\int_{\mathscr{L}} P(x,u)\dot{u}=1$;

(iii) 当 x 趋于边界点 v 时,

$$\lim_{x\to v} P(x,u)=\begin{cases} 0, & 若\ u\neq v, \\ \infty, & 若\ u=v; \end{cases}$$

(iv) 它适合一个线性算子 (不一定是微分算子)

$$\partial\Phi=0. \tag{A}$$

则我们可以希望由

$$\Phi(x)=\int_{\mathscr{L}} P(x,u)\Phi(u)\dot{u}$$

来给出方程 (A) 的 Dirichlet 问题的解答来, 例如在单位球内

$$\frac{1-xx'}{(1-2ux'+xx')^{n/2}}$$

也具有以下的一些性质:

(1) 非负性;

(2), (3) "δ" 函数性;

(4) 它适合于普通的 Laplace 方程

$$\sum_{i=1}^{n}\frac{\partial^2\Phi}{\partial x_i^2}=0.$$

因此 Poisson 积分

$$\Phi(x)=\frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}\frac{1-xx'}{(1-2ux'+xx')^{n/2}}\Phi(u)\dot{u}$$

也自然地给出了单位球内的 Dirichlet 问题的解.

以上的性质中较难证明的只有两点:

(i) $\dfrac{1}{\omega_{n-1}}\displaystyle\int\underset{uu'=1}{\cdots}\int\frac{1-xx'}{(1-2ux'+xx')^{n/2}}\dot{u}=1;$

(ii) $\displaystyle\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left[\frac{1-xx'}{(1-2ux'+xx')^{n/2}}\right]=0.$

附记　在《典型域的调和分析》一书中出现了更多更复杂的类型的“δ 函数”

1.9　对 称 原 理

对称原理的重要根据之一是: 二维的 Laplace 方程

$$\left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\partial^2}{\partial\theta^2}\right]\varPhi=0$$

经反演而不变, 即命

$$\tau=\frac{1}{\rho},$$

则

$$\rho\frac{\partial}{\partial\rho}=-\tau\frac{\partial}{\partial\tau}.$$

但当 $n\geqslant 3$ 时, 这一性质不再成立.

Laplace 方程

$$\frac{1}{\rho^{n-3}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}\right)+\partial_u^2=\rho^2\frac{\partial^2}{\partial\rho^2}+(n-1)\rho\frac{\partial}{\partial\rho}+\partial_u^2$$

变为

$$\begin{aligned}&\frac{1}{\rho^{n-2}}\left(\rho\frac{\partial}{\partial\rho}\right)\left(\rho^{n-2}\rho\frac{\partial}{\partial\rho}\right)+\partial_u^2\\=&\tau^{n-2}\left(\tau\frac{\partial}{\partial\tau}\right)\left(\frac{1}{\tau^{n-2}}\tau\frac{\partial}{\partial\tau}\right)+\partial_u^2\\=&\tau^{n-1}\frac{\partial}{\partial\tau}\left(\tau^{-n+3}\frac{\partial}{\partial\tau}\right)+\partial_u^2\\=&\tau^2\frac{\partial^2}{\partial\tau^2}-(n-3)\tau\frac{\partial}{\partial\tau}+\partial_u^2,\end{aligned}$$

也就是 Laplace 方程经反演而变了. 幸而有, 如果把被微分函数 $\varPhi$ 改为 $\tau^{n-2}\varPsi$, 则

$$\begin{aligned}\frac{\partial\varPhi}{\partial\tau}&=\tau^{n-2}\frac{\partial\varPsi}{\partial\tau}+(n-2)\tau^{n-3}\varPsi,\\\frac{\partial^2\varPhi}{\partial\tau^2}&=\tau^{n-2}\frac{\partial^2\varPsi}{\partial\tau^2}+2(n-2)\tau^{n-3}\frac{\partial\varPsi}{\partial\tau}+(n-2)(n-3)\tau^{n-4}\varPsi.\end{aligned}$$

因而

$$\left[\tau^2\frac{\partial^2}{\partial\tau^2}-(n-3)\tau\frac{\partial}{\partial\tau}+\partial_u^2\right]\Phi=\tau^{n-2}\left[\tau^2\frac{\partial^2\Psi}{\partial\tau^2}+(n-1)\tau\frac{\partial\Psi}{\partial\tau}\right]+\tau^{n-2}\partial_u^2\Psi,$$

即 Laplace 方程如果变数 x 和函数 Φ 都经过变化

$$y=\frac{x}{xx'},\quad \Psi(y)=(xx')^{\frac{n}{2}-1}\Phi(x)$$

则也不变.

换言之, 在研究 n 维 Laplace 方程的对称原理时, 必须注意: 变数 x 与函数 Φ 都要经过变换, 但对我们所研究的微分方程

$$\frac{(1-\rho^2)^n}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\frac{\rho^{n-1}}{(1-\rho^2)^{n-2}}\frac{\partial\Phi}{\partial\rho}\right)+\frac{(1-\rho^2)^2}{\rho^2}\partial_u^2\Phi=0$$

来说, 它经 $\rho=\dfrac{1}{\tau}$ 而不变的, 因而可如两个变数的办法直接推广.

1.10 Laplace 方程的不变性

Laplace 方程虽然经过

$$y=\frac{(1-aa')(x-a)-(x-a)(x-a)'a}{1-2ax'+aa'xx'}\quad (aa'<1) \tag{1}$$

而改变, 但是如果被微分的函数也相应地发生变化, 我们也可以找出另一些不变性, 就是如果自变数照 (1) 变化, 而函数照

$$Y=\left(\frac{1-2ax'+aa'xx'}{1-aa'}\right)^{\frac{n}{2}-1}X \tag{2}$$

变化, 我们有

$$(1-xx')^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2X}{\partial x_i^2}=(1-yy')^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2Y}{\partial y_i^2}. \tag{3}$$

在证明此式之前, 先直接验算以下的 $n+1$ 个函数适合 Laplace 方程:

$$\Phi(x)=(1-2ax'+aa'xx')^{1-\frac{n}{2}} \tag{4}$$

及

$$\Psi(x)=(1-2ax'+aa'xx')^{-\frac{n}{2}}[(1-aa')(x-a)-(x-a)(x-a)'a] \tag{5}$$

((5) 是一矢量, 共有 n 个函数).

先证 $\Phi(x)$ 是调和函数:

$$\begin{aligned}\frac{\partial \Phi}{\partial x_i} =&2\left(1-\frac{n}{2}\right)(1-2ax'+aa'xx')^{-\frac{n}{2}}(aa'x_i-a_i),\\ \sum_{i=1}^{n}\frac{\partial^2 \Phi}{\partial x_i^2} =&4\left(1-\frac{n}{2}\right)\left(-\frac{n}{2}\right)(1-2ax'+aa'xx')^{-\frac{n}{2}-1}\\ &\times\sum_{i=1}^{n}(aa'x_i-a_i)^2+2\left(1-\frac{n}{2}\right)(1-2ax'+aa'xx')^{-\frac{n}{2}}naa'\\ =&0.\end{aligned}$$

再证 $\Psi(x)$ 是调和函数:

$$\begin{aligned}\frac{\partial \Psi}{\partial x_i} =&-n(1-2ax'+aa'xx')^{-\frac{n}{2}-1}(aa'x_i-a_i)\\ &\times[(1-aa')(x-a)-(x-a)(x-a)'a]\\ &+(1-2ax'+aa'xx')^{-\frac{n}{2}}[(1-aa')e_i-2(x_i-a_i)a],\end{aligned}$$

这里 $e_i=(0,0,\cdots,0,1,0,\cdots,0)$, 第 i 支量为 1, 其他为 0. 又

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial x_i^2} =&n(n+2)(1-2ax'+aa'xx')^{-\frac{n}{2}-2}\\ &\times(aa'x_i-a_i)^2[(1-aa')(x-a)-(x-a)(x-a)'a]\\ &-n(1-2ax'+aa'xx')^{-\frac{n}{2}-1}aa'[(1-aa')(x-a)\\ &-(x-a)(x-a)'a]-2n(1-2ax'+aa'xx')^{-\frac{n}{2}-1}\\ &\times(aa'x_i-a_i)[(1-aa')e_i-2(x_i-a_i)a]\\ &+(1-2ax'+aa'xx')^{-\frac{n}{2}}(-2a).\end{aligned}$$

因此

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 \Psi}{\partial x_i^2} =&n(n+2)(1-2ax'+aa'xx')^{-\frac{n}{2}-1}\\ &\times aa'[(1-aa')(x-a)-(x-a)(x-a)'a]\\ &-n^2(1-2ax'+aa'xx')^{-\frac{n}{2}-1}aa'[(1-aa')(x-a)\\ &-(x-a)(x-a')a]-2n(1-2ax'+aa'xx')^{-\frac{n}{2}-1}\\ &\times[aa'(1-aa')x-(1-aa')a-2aa'(xx'-ax)a\\ &+2(ax'-aa')a]-2n(1-2ax'+aa'xx')^{-\frac{n}{2}}a\\ =&0.\end{aligned}$$

现在来证明 (3) 式, 先求

$$X = \Phi(x)Y(1-aa')^{\frac{n}{2}-1}$$

的偏微商:

$$(1-aa')^{1-\frac{n}{2}}\frac{\partial X}{\partial x_i} = \sum_{j=1}^{n}\frac{\partial Y}{\partial y_j}\frac{\partial y_j}{\partial x_i}\Phi(x) + \frac{\partial \Phi}{\partial x_i}Y,$$

$$(1-aa')^{1-\frac{n}{2}}\frac{\partial^2 X}{\partial x_i^2} = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial y_j\partial y_k}\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\Phi(x)$$
$$+\sum_{j=1}^{n}\frac{\partial Y}{\partial y_j}\frac{\partial^2 y_j}{\partial x_i^2}\Phi(x) + 2\sum_{j=1}^{n}\frac{\partial Y}{\partial y_j}\frac{\partial y_j}{\partial x_i}\frac{\partial \Phi}{\partial x_i} + \frac{\partial^2 \Phi}{\partial x_i^2}Y,$$

$$(1-aa')^{1-\frac{n}{2}}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2} = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial y_j\partial y_k}\sum_{i=1}^{n}\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\Phi(x)$$
$$+\sum_{j=1}^{n}\frac{\partial Y}{\partial y_j}\sum_{i=1}^{n}\left(\frac{\partial^2 y_j}{\partial x_i^2}\Phi(x) + 2\frac{\partial y_j}{\partial x_i}\frac{\partial \Phi}{\partial x_i}\right) + \sum_{i=1}^{n}\frac{\partial^2 \Phi}{\partial x_i^2}Y$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 Y}{\partial y_j\partial y_k}\sum_{i=1}^{n}\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i}\Phi(x) + \sum_{j=1}^{n}\frac{\partial Y}{\partial y_j}$$
$$\times\left[\sum_{i=1}^{n}\frac{\partial^2 \Psi_j(x)}{\partial x_i^2} - \sum_{i=1}^{n}\frac{\partial^2 \Phi}{\partial x_i^2}y_j\right] + \sum_{i=1}^{n}\frac{\partial^2 \Phi}{\partial x_i^2}Y,$$

其后二项都等于 0.

又由

$$dydy' = \frac{(1-aa')^2}{(1-2ax'+aa'xx')^2}dxdx',$$

推得

$$\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_j}\frac{\partial y_i}{\partial x_k} = \frac{(1-aa')^2}{(1-2ax'+aa'xx')^2}\delta_{jk},$$

乘以 $\dfrac{\partial x_k}{\partial y_l}$ 对 k 加之, 得

$$\frac{(1-aa')^2}{(1-2ax'+aa'xx')^2}\frac{\partial x_j}{\partial y_l} = \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_j}\frac{\partial y_i}{\partial x_k}\right)\frac{\partial x_k}{\partial y_l} = \frac{\partial y_l}{\partial x_j},$$

因此

$$\sum_{i=1}^{n}\frac{\partial y_j}{\partial x_i}\frac{\partial y_k}{\partial x_i} = \sum_{i=1}^{n}\frac{\partial y_j}{\partial x_i}\frac{(1-aa')^2}{(1-2ax'+aa'xx')^2}\frac{\partial x_i}{\partial y_k}$$

$$=\frac{(1-aa')^2}{(1-2ax'+aa'xx')^2}\delta_{jk}.$$

由 (6) 推出

$$(1-aa')^{1-\frac{n}{2}}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}=\frac{(1-aa')^2\Phi(x)}{(1-2ax'+aa'xx')^2}\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2}$$

$$=(1-aa')^2(1-2ax'+aa'xx')^{-\frac{n}{2}-1}\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2},\qquad(7)$$

利用关系

$$1-xx'=(1-2ax'+aa'xx')(1-yy')/(1-aa')$$

可得

$$(1-xx')^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}=(1-yy')^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2}.$$

附记　(1) 实质上对所有的 Möbius 变换

$$w=\frac{az+b}{cz+d},\quad ad-bc=1$$

都有

$$dw=\frac{dz}{(cz+d)^2},$$

因此

$$dwd\overline{w}=\frac{dzd\overline{z}}{|cz+d|^4}$$

及

$$\partial_w\partial_{\overline{w}}\Phi=|cz+d|^4\partial_z\partial_{\overline{z}}\Phi,$$

即可望得出更一般的结果. 由于 Möbius 变换的实形式的 n 维推广是共形变换, 也就是球几何!

(2) 对称原理最简单的形式是把 R_1(图 1) 中的调和函数 (在实数轴上的条件略) 可以扩展到 R_2. 而直线与任何圆等价, 所以一般的形式是任何一段圆弧都有此对应的结果. 推广到 n 维, 任何一个球的任何一片都可应用对称原理解析扩展出去.

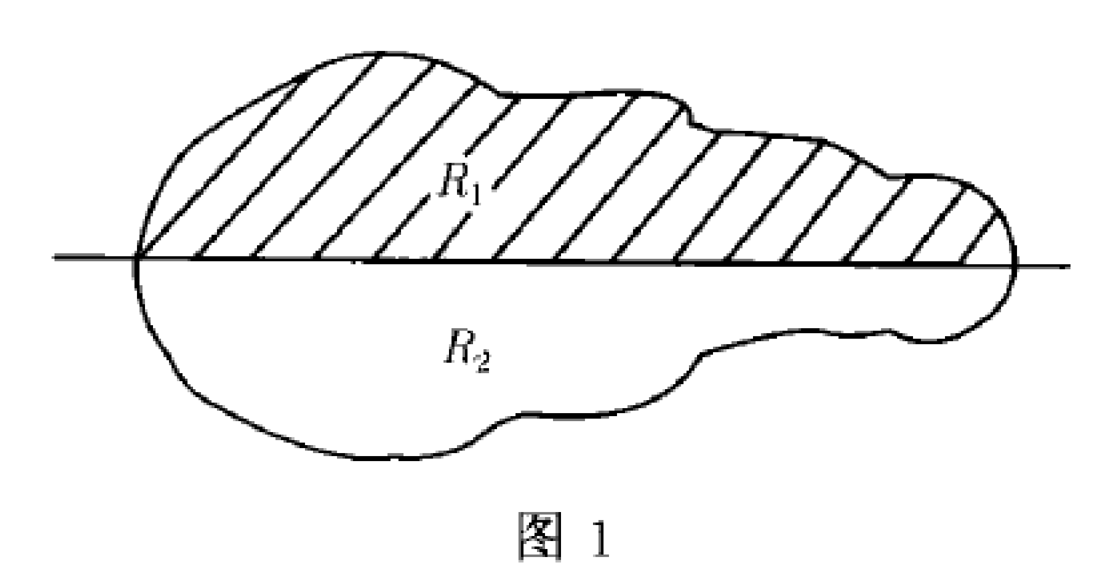


图 1

1.5 节中微分方程 (1) 经反演

$$y = \frac{x}{xx'}$$

而不变, 对称原理获得自然推广. 用第 3 讲 3.6 节中的结果将获得更普遍的形式.

1.11 Laplace 方程的均值公式

定理 1 如果 $\Phi(x) = \Phi(\rho u)$ 是在单位球上 $xx' \leqslant 1$ 适合于 Laplace 方程

$$\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = \frac{1}{\rho^{n-1}} \frac{\partial}{\partial \rho} \left(\rho^{n-1} \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \partial_u^2 \Phi = 0, \tag{1}$$

而且有二阶连续偏微商的函数, 则当 $0 \leqslant \rho \leqslant 1$ 时

$$\frac{1}{\omega_{n-1}} \int \cdots \int_{uu'=1} \Phi(\rho u) \dot{u} = \Phi(0). \tag{2}$$

命

$$F(\rho) = \frac{1}{\omega_{n-1}} \int \cdots \int_{uu'=1} \Phi(\rho u) \dot{u}$$

积分号下求微分, 并由 (1) 可知

$$\begin{aligned} \frac{1}{\rho^{n-3}} \frac{d}{d\rho} \left(\rho^{n-1} \frac{dF}{d\rho} \right) &= \frac{1}{\omega_{n-1}} \int \cdots \int_{uu'=1} \frac{1}{\rho^{n-3}} \frac{\partial}{\partial \rho} \left(\rho^{n-1} \frac{\partial \Phi}{\partial \rho} \right) \dot{u} \\ &= -\frac{1}{\omega_{n-1}} \int \cdots \int_{uu'=1} \partial_u^2 \Phi \dot{u}. \end{aligned}$$

由

$$\begin{aligned} \partial_u^2 &= \sum_{p=1}^{n-1} \left(\frac{1}{\sin^2 \theta_1 \cdots \sin \theta_{p-1}} \frac{\partial^2}{\partial \theta_p^2} + (n-p-1) \frac{\cot\theta_p}{\sin^2 \theta_1 \cdots \sin^2 \theta_{p-1}} \frac{\partial}{\partial \theta_p} \right) \\ &= \sum_{p=1}^{n-1} \frac{1}{\sin^2 \theta_1 \cdots \sin \theta_{p-1}} \cdot \frac{1}{\sin^{n-p-1} \theta_p} \frac{\partial}{\partial \theta_p} \left(\sin^{n-p-1} \theta_p \frac{\partial}{\partial \theta_p} \right) \end{aligned}$$

及

$$\dot{u} = \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \cdots \sin^{n-p-1} \theta_p \cdots \sin \theta_{n-2} d\theta_1 \cdots d\theta_{n-1},$$

可知

$$\int \cdots \int_{uu'=1} \partial_u^2 \Phi(\rho u) \dot{u} = \sum_{p=1}^{n-1} J_p.$$

此处

$$J_p = \int_{uu'=1}\cdots\int \frac{\partial}{\partial\theta_p}\left(\sin^{n-p-1}\theta_p \frac{\partial\Phi}{\partial\theta_p}\right) \frac{\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}d\theta_1\cdots d\theta_{n-1}}{\sin^2\theta_1\cdots\sin^2\theta_{p-1}\sin^{n-p-1}\theta_p}.$$

J_p 是一个 $(n-1)$ 重积分, 其中第 $p(p<n-1)$ 重积分是

$$\int_0^{\pi} \frac{\partial}{\partial\theta_p}\left(\sin^{n-p-1}\theta_p \frac{\partial\Phi}{\partial\theta_p}\right) d\theta_p = \sin^{n-p-1}\theta_p \left.\frac{\partial\Phi}{\partial\theta_p}\right|_0^{\pi} = 0.$$

因此当 $p<n-1$ 时, $J_p=0$, 当 $p=n-1$ 时, J_p 中第 $(n-1)$ 重积分是

$$\int_0^{2\pi} \frac{\partial^2\Phi}{\partial\theta_{n-1}^2} d\theta_{n-1} = \left.\frac{\partial\Phi}{\partial\theta_{n-1}}\right|_0^{2\pi} = 0,$$

这里用了另外一个性质, Φ 对 θ_{n-1} 的周期性, 因此 $J_{n-1}=0$. 总之

$$\int_{uu'=1}\cdots\int \partial_u^2\Phi\dot{u} = 0,$$

因而

$$\frac{1}{\rho^{n-3}}\frac{d}{d\rho}\left(\rho^{n-1}\frac{dF}{d\rho}\right) = 0,$$

推得

$$\rho^{n-1}\frac{dF}{d\rho} = k$$

为常数, 当 $\rho=0$ 时可见 $k=0$, 因此 F 是一常数, 再取 $\rho=0$, 得出公式 (2).

1.12　Laplace 方程的 Poisson 公式

回到 1.10 节, 命

$$Y(y) = \left(\frac{1-2ax'+aa'xx'}{1-aa'}\right)^{\frac{n}{2}-1} X(x),$$

如果 $Y(y)$ 适合于 Laplace 方程, 则 $X(x)$ 也对. 对 $Y(y)$ 用均值公式

$$\frac{1}{\omega_{n-1}}\int_{vv'=1}\cdots\int Y(v)\dot{v} = Y(0).$$

由于

$$Y(0) = \left(\frac{1-2aa'+(aa')^2}{1-aa'}\right)^{\frac{n}{2}-1} X(a) = (1-aa')^{\frac{n}{2}-1}X(a),$$

$$Y(v)=\left(\frac{1-2au'+aa'}{1-aa'}\right)^{\frac{n}{2}-1}X(u)$$

及

$$\dot{v}=\left(\frac{1-aa'}{1-2au'+aa'}\right)^{n-1}\dot{u},$$

因此得出

$$\begin{aligned}(1-aa')^{\frac{n}{2}-1}X(a)=&Y(0)=\frac{1}{\omega_{n-1}}\int\cdots\int_{vv'=1}Y(v)\dot{v}\\ &=\frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}X(u)\left(\frac{1-2au'+aa'}{1-aa'}\right)^{\frac{n}{2}-1}\\ &\quad\times\left(\frac{1-aa'}{1-2au'+aa'}\right)^{n-1}\dot{u}.\end{aligned}$$

于是得到 Laplace 方程的 Poisson 公式

$$X(a)=\frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}\frac{1-aa'}{(1-2au'+aa')^{n/2}}X(u)\dot{u}. \tag{1}$$

这给出了 Laplace 方程的 Dirichlet 问题的解的唯一性.

1.13 小 结

从单位圆出发, 直接推广到单位球

$$xx'<1. \tag{1}$$

这单位球的边界是

$$uu'=1. \tag{2}$$

单位球的表面积元素以 $\dot{u}$ 表之, 总表面积 $\int\cdots\int_{uu'=1}\dot{u}=\omega_{n-1}$.

本讲中提出以下几种看法:

(1) 从单位圆的实形式出发.

变换群: 由把 $x=a$ 变为 $y=0$ 的变换

$$y=\frac{(1-aa')(x-a)-(x-a)(x-a)'a}{1-2ax'+aa'xx'},\quad aa'<1 \tag{3}$$

及使 0 点不变的变换

$$y=x\Gamma,\quad \Gamma\Gamma'=I \tag{4}$$

所组成. 由此推出

$$1-yy'=\frac{(1-aa')(1-xx')}{1-2ax'+aa'xx'}. \tag{5}$$

这显示出把 (1) 变为其自己. 微分不变量是

$$\frac{dydy'}{(1-yy')^2}=\frac{dxdx'}{(1-xx')^2}. \tag{6}$$

二阶偏微分不变算子是

$$\begin{aligned}&(1-yy')^n\sum_{i=1}^n\frac{\partial}{\partial y_i}\left((1-yy')^{2-n}\frac{\partial u}{\partial y_i}\right)\\ =&(1-xx')^n\sum_{i=1}^n\frac{\partial}{\partial x_i}\left((1-xx')^{2-n}\frac{\partial u}{\partial x_i}\right)\end{aligned}$$

因而有二阶偏微分方程式

$$(1-xx')^n\sum_{i=1}^n\frac{\partial}{\partial x_i}\left((1-xx')^{2-n}\frac{\partial u}{\partial x_i}\right)=0. \tag{7}$$

变换 (3)、(4) 也使 (2) 不变. 在 (2) 上体积元素的变化规律是

$$\dot{v}=\left(\frac{1-aa'}{1-2au'+aa'}\right)^{n-1}\dot{u}. \tag{8}$$

这里得出了 Poisson 核. 由此可有 Poisson 公式

$$\Phi(x)=\frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}\left(\frac{1-xx'}{1-2xu'+xx'}\right)^{n-1}\Phi(u)\dot{u}. \tag{9}$$

来解决 (7) 式的边界值问题 (Dirichlet 问题).

(2) 从实射影群出发.

变换群由把 $x=a$ 变为 $y=0$ 的变换

$$y=\frac{\sqrt{1-aa'}(x-a)(I+\lambda a'a)}{1-ax'},\quad \lambda=\frac{1-\sqrt{1-aa'}}{aa'\sqrt{1-aa'}} \tag{3'}$$

及使 0 点不变的变换

$$y=x\Gamma,\quad \Gamma\Gamma'=I \tag{4'}$$

所组成. 由此推出

$$1-yy'=\frac{(1-aa')(1-xx')}{(1-ax')^2}, \tag{5'}$$

微分不变量是

$$\frac{dy(I-y'y)^{-1}dy'}{1-yy'}=\frac{dx(I-x'x)^{-1}dx'}{1-xx'}. \tag{6'}$$

由不变二阶偏微分算子所得出的方程是

$$\sum_{i=1}^{n}\frac{\partial^2\Phi}{\partial x_i^2}-\sum_{i,j=1}^{n}x_ix_j\frac{\partial^2\Phi}{\partial x_i\partial x_j}-2\sum_{i=1}^{n}x_i\frac{\partial\Phi}{\partial x_i}=0. \tag{7'}$$

在 (2) 上体积函数的变化规律是

$$\dot{v}=\frac{(1-aa')^{\frac{1}{2}(n-1)}}{(1-au')^{n-1}}\dot{u}. \tag{8'}$$

因此解决 (7′) 的边界值问题的 Poisson 公式是

$$\Phi(x)=\frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}\frac{(1-xx')^{\frac{1}{2}(n-1)}}{(1-xu')^{n-1}}\varphi(u)\dot{u}. \tag{9'}$$

(3) 在 (1) 中所讨论的几何学为第 3 讲的球几何共形映照作了准备. 而在 (2) 中所讨论的几何学则为混合型偏微分方程作准备. (7′) 的二次型是

$$I-x'x.$$

当 $xx'<1$ 时, 这方阵是定正的; 而 $xx'>1$ 时, 它是一负 $(n-1)$ 正的, 由于 (2) 的变形是一次的, 因此在反演下不是不变的.

(4) Laplace 算子虽然经过 (3) 而改变, 但如果考虑其被作用的函数也变化, 我们仍能得出共变形式：当 x,y 经过 (3) 变化时, 我们有

$$Y=\left(\frac{1-2ax'+aa'xx'}{1-aa'}\right)^{\frac{n}{2}-1}X. \tag{10}$$

则

$$(1-xx')^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2X}{\partial x_i^2}=(1-yy')^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2Y}{\partial y_i^2}. \tag{7''}$$

1.12 节中给出了 Laplace 方程的 Poisson 公式

$$X(x)=\frac{1}{\omega_{n-1}}\int\cdots\int_{uu'=1}\frac{1-xx'}{(1-2xu'+xx')^{\frac{n}{2}}}X(u)\dot{u}.$$

(5) 单位圆推广到单位球是开始的第一步. 关于部分的发展可以参阅《多复变函数论中典型域的调和分析》

第 2 讲　Fourier 分析与调和函数的展开式

2.1　超球函数的一些性质

为了便于了解起见，我们从头起叙述超球多项式的一些性质：

当 $\lambda > -\frac{1}{2}$ 时，超球多项式由

$$P_m^{(\lambda)}(\xi) = \sum_{0 \leqslant l \leqslant \frac{1}{2}m} (-1)^l \frac{\Gamma(m-l+\lambda)}{\Gamma(\lambda) l!(m-2l)!}(2\xi)^{m-2l} \tag{1}$$

来定义，它是一 m 次的多项式，有时还定义 $P_{-1}^{(\lambda)}(\xi) = 0$. 不难算出

$$\begin{aligned}
P_0^{(\lambda)}(\xi) =& 1, \quad P_1^{(\lambda)}(\xi) = 2\lambda\xi, \\
P_2^{(\lambda)}(\xi) =& 2\lambda(\lambda+1)\xi^2 - \lambda, \\
P_3^{(\lambda)}(\xi) =& \frac{4}{3}\lambda(\lambda+1)(\lambda+2)\xi^3 - 2\lambda(\lambda+1)\xi, \\
& \cdots\cdots
\end{aligned}$$

一般可以从递归公式

$$mP_m^{(\lambda)}(\xi) = 2(m+\lambda-1)\xi P_{m-1}^{(\lambda)}(\xi) - (m+2\lambda-2)P_{m-2}^{(\lambda)}(\xi) \tag{2}$$

逐一推出，(2) 式右边等于

$$\begin{aligned}
&(m+\lambda-1) \sum_{0 \leqslant l \leqslant \frac{1}{2}(m-1)} (-1)^l \frac{\Gamma(m-1-l+\lambda)}{\Gamma(\lambda) l!(m-1-2l)!}(2\xi)^{m-2l} \\
&- (m+2\lambda-2) \sum_{0 \leqslant l \leqslant \frac{1}{2}(m-2)} (-1)^l \frac{\Gamma(m-2-l+\lambda)}{\Gamma(\lambda) l!(m-2-2l)!} \times (2\xi)^{m-2l-2} \\
=& \sum_{0 \leqslant l \leqslant \frac{1}{2}m} (-1)^l \left[\frac{\Gamma(m-1-l+\lambda)(m+\lambda-1)}{\Gamma(\lambda) l!(m-1-2l)!}\right. \\
&\left. + \frac{\Gamma(m-1-l+\lambda)(m+2\lambda-2)}{\Gamma(\lambda)(l-1)!(m-2l)!}\right](2\xi)^{m-2l} \\
=& \sum_{0 \leqslant l \leqslant \frac{1}{2}m} (-1)^l \frac{\Gamma(m-1-l+\lambda)}{\Gamma(\lambda) l!(m-2l)!} \\
&\times [(m+\lambda-1)(m-2l) + (m+2\lambda-2)l](2\xi)^{m-2l}
\end{aligned}$$

$$=m\sum_{0\leqslant l\leqslant\frac{1}{2}m}(-1)^l\frac{\Gamma(m-l+\lambda)}{\Gamma(\lambda)l!(m-2l)!}(2\xi)^{m-2l}.$$

即得所证.

从递归公式 (2), 用归纳法立刻证得

$$\sum_{m=0}^{n}(\lambda+m)P_m^{(\lambda)}(\xi)=\frac{1}{2}\frac{(n+2\lambda)P_n^{(\lambda)}(\xi)-(n+1)P_{n+1}^{(\lambda)}(\xi)}{1-\xi}. \tag{3}$$

乘 (2) 式以 ρ^{m-1}, 而对 m 相加, 得到

$$\begin{aligned}\sum_{m=0}^{\infty}m\rho^{m-1}P_m^{(\lambda)}(\xi)=&2\sum_{m=0}^{\infty}(m+\lambda-1)\xi P_{m-1}^{(\lambda)}(\xi)\rho^{m-1}\\&-\sum_{m=0}^{\infty}(m+2\lambda-2)P_{m-2}^{(\lambda)}(\xi)\rho^{m-1}.\end{aligned}$$

命

$$h(\rho)=\sum_{m=0}^{\infty}P_m^{(\lambda)}(\xi)\rho^m.$$

则此式可以写成为

$$\begin{aligned}h'(\rho)=&2\xi\rho^{1-\lambda}[\rho^\lambda h(\rho)]'-\rho^{2-2\lambda}[\rho^{2\lambda}h(\rho)]'\\=&2\xi[\lambda h(\rho)+\rho h'(\rho)]-[2\lambda\rho h(\rho)+\rho^2h'(\rho)],\end{aligned}$$

或

$$\frac{h'(\rho)}{h(\rho)}=2\lambda(\xi-\rho)/(1-2\xi\rho+\rho^2).$$

运用 $h(0)=P_0^{(\lambda)}(\xi)=1$, 积分此式立得

$$h(\rho)=(1-2\xi\rho+\rho^2)^{-\lambda}.$$

也就是我们有演出函数

$$(1-2\xi\rho+\rho^2)^{-\lambda}=\sum_{m=0}^{\infty}P_m^{(\lambda)}(\xi)\rho^m. \tag{4}$$

微分公式 (1) 得

$$\begin{aligned}\frac{d}{d\xi}P_m^{(\lambda)}(\xi)=&2\sum_{0\leqslant l\leqslant\frac{1}{2}(m-1)}(-1)^l\frac{\Gamma(m-l+\lambda)}{\Gamma(\lambda)l!(m-2l-1)!}\times(2\xi)^{m-1-2l}\\=&2\lambda\sum_{0\leqslant l\leqslant\frac{1}{2}(m-1)}(-1)^l\frac{\Gamma(m-1-l+\lambda+1)}{\Gamma(\lambda+1)l!(m-1-2l)!}(2\xi)^{m-1-2l}.\end{aligned}$$

故得微分递归公式:

$$\frac{d}{d\xi}P_m^{(\lambda)}(\xi)=2\lambda P_{m-1}^{(\lambda+1)}(\xi). \tag{5}$$

也不难证明:

$$(1-\xi^2)\frac{d^2}{d\xi^2}P_m^{(\lambda)}(\xi)-(2\lambda+1)\xi\frac{d}{d\xi}P_m^{(\lambda)}(\xi)+m(m+2\lambda)P_m^{(\lambda)}(\xi)=0.$$

如命

$$\eta=(1-\xi^2)^{\lambda-\frac{1}{2}}P_m^{(\lambda)}(\xi), \tag{6}$$

则 η 适合于微分方程

$$(1-\xi^2)\frac{d^2\eta}{d\xi^2}+(2\lambda-3)\xi\frac{d\eta}{d\xi}+(m+1)(m+2\lambda-1)\eta=0. \tag{7}$$

今往证明 Rodrique 公式:

$$(1-\xi^2)^{\lambda-\frac{1}{2}}P_m^{(\lambda)}(\xi)=\frac{(-2)^m}{m!}\frac{\Gamma(m+\lambda)\Gamma(m+2\lambda)}{\Gamma(\lambda)\Gamma(2m+2\lambda)}\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{m+\lambda-\frac{1}{2}}. \tag{8}$$

在证明此公式之前, 先由 (1) 推得恒等式

$$mP_m^{(\lambda)}(\xi)=(m+2\lambda-1)\xi P_{m-1}^{(\lambda)}(\xi)-2\lambda(1-\xi^2)P_{m-2}^{(\lambda+1)}(\xi), \tag{9}$$

再行归纳法, 此式之右边等于

$$\begin{aligned}
&(m+2\lambda-1)\xi(1-\xi^2)^{-\lambda+\frac{1}{2}}\frac{(-2)^{m-1}}{(m-1)!}\frac{\Gamma(m-1+\lambda)}{\Gamma(\lambda)}\\
&\times\frac{\Gamma(m-1+2\lambda)}{\Gamma(2m+2\lambda-2)}\left(\frac{d}{d\xi}\right)^{m-1}(1-\xi^2)^{m+\lambda-\frac{3}{2}}\\
&-2\lambda(1-\xi^2)^{-\lambda+\frac{1}{2}}\frac{(-2)^{m-2}}{(m-2)!}\frac{\Gamma(m-1+\lambda)}{\Gamma(\lambda+1)}\\
&\times\frac{\Gamma(m+2\lambda)}{\Gamma(2m+2\lambda-2)}\left(\frac{d}{d\xi}\right)^{m-2}(1-\xi^2)^{m+\lambda-\frac{3}{2}}\\
=&\frac{(-2)^{m-1}\Gamma(m-1+\lambda)\Gamma(m+2\lambda)}{(m-1)!\Gamma(\lambda)\Gamma(2m+2\lambda-2)}(1-\xi^2)^{-\lambda+\frac{1}{2}}\\
&\times\left[\xi\left(\frac{d}{d\xi}\right)^{m-1}(1-\xi^2)^{m+\lambda-\frac{3}{2}}+(m-1)\left(\frac{d}{d\xi}\right)^{m-2}(1-\xi^2)^{m+\lambda-\frac{3}{2}}\right]\\
=&\frac{(-2)^{m-1}\Gamma(m-1+\lambda)\Gamma(m+2\lambda)}{(m-1)!\Gamma(\lambda)\Gamma(2m+2\lambda-2)}(1-\xi^2)^{-\lambda+\frac{1}{2}}\\
&\times\left(\frac{d}{d\xi}\right)^{m-1}\left\{(1-\xi^2)^{m+\lambda-\frac{3}{2}}\xi\right\}\text{(由 Leibnitz 公式)}
\end{aligned}$$

$$=\frac{(-2)^{m-1}\Gamma(m-1+\lambda)\Gamma(m+2\lambda)}{(m-1)!\Gamma(\lambda)\Gamma(2m+2\lambda-2)}\left(\frac{-1}{2m+2\lambda-1}\right)$$
$$\times(1-\xi^2)^{-\lambda+\frac{1}{2}}\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{m+\lambda-\frac{1}{2}}$$
$$=\frac{-(-2)^{m-1}\Gamma(m-1+\lambda)\Gamma(m+2\lambda)(2m+2\lambda-2)}{(m-1)!\Gamma(\lambda)\Gamma(2m+2\lambda-2)\cdot(2m+2\lambda-2)(2m+2\lambda-1)}$$
$$\times(1-\xi^2)^{-\lambda+\frac{1}{2}}\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{m+\lambda-\frac{1}{2}}$$
$$=\frac{(-2)^m\Gamma(m+\lambda)\Gamma(m+2\lambda)}{(m-1)!\Gamma(\lambda)\Gamma(2m+2\lambda)}(1-\xi^2)^{-\lambda+\frac{1}{2}}\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{m+\lambda-\frac{1}{2}}.$$

2.2 正交性质

假定 $f(\xi)$ 是一个在 $[-1,+1]$ 之间有 m 次连续微商的函数, 由 Rodrique 公式可知

$$\int_{-1}^{1}f(\xi)P_m^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}d\xi$$
$$=\frac{(-2)^m\Gamma(m+\lambda)\Gamma(m+2\lambda)}{m!\Gamma(\lambda)\Gamma(2m+2\lambda)}\int_{-1}^{1}f(\xi)\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi. \tag{1}$$

用分部积分可知

$$\int_{-1}^{1}f(\xi)\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi$$
$$=f(\xi)\left(\frac{d}{d\xi}\right)^{m-1}(1-\xi^2)^{m+\lambda-\frac{1}{2}}\Big|_{-1}^{1}$$
$$-\int_{-1}^{1}f'(\xi)\left(\frac{d}{d\xi}\right)^{m-1}(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi.$$

由于 $\lambda>-\dfrac{1}{2}$, 所以

$$\left(\frac{d}{d\xi}\right)^{m-1}(1-\xi^2)^{m+\lambda-\frac{1}{2}}\Big|_{-1}^{1}=0.$$

因此得出

$$\int_{-1}^{1}f(\xi)\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi$$
$$=-\int_{-1}^{1}f'(\xi)\left(\frac{d}{d\xi}\right)^{m-1}(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi.$$

续行此法, 最后得出

$$\int_{-1}^{1} f(\xi)P_m^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}d\xi = \frac{2^m\Gamma(m+\lambda)\Gamma(m+2\lambda)}{m!\Gamma(\lambda)\Gamma(2m+2\lambda)} \times \int_{-1}^{1}(1-\xi^2)^{m+\lambda-\frac{1}{2}}\left(\frac{d}{d\xi}\right)^m f(\xi)d\xi. \qquad (2)$$

如果 $f(\xi)$ 是一 m 次多项式, 其最高方次的系数等于 a, 则得

$$\begin{aligned}\int_{-1}^{1} f(\xi)P_m^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}d\xi &= \frac{2^m\Gamma(m+\lambda)\Gamma(m+2\lambda)}{\Gamma(\lambda)\Gamma(2m+2\lambda)}a\int_{-1}^{1}(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi \\ &= \frac{2^m\Gamma(m+\lambda)\Gamma(m+2\lambda)\Gamma\left(m+\lambda+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma(2m+2\lambda)\Gamma(m+\lambda+1)}a \\ &= \frac{2^{-m-2\lambda+1}\pi\Gamma(m+2\lambda)}{\Gamma(\lambda)\Gamma(m+\lambda+1)}a, \qquad (3)\end{aligned}$$

此处用了公式

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\pi^{\frac{1}{2}}\Gamma(2x).$$

特别取 $f(\xi)=P_l^{(\lambda)}(\xi)$, 由 (1.1) 可知

$$a = \begin{cases} 0, & 若\ l<m, \\ 2^m\dfrac{\Gamma(m+\lambda)}{\Gamma(\lambda)m!}, & 若\ l=m, \end{cases}$$

则得

$$\int_{-1}^{1} P_l^{(\lambda)}(\xi)P_m^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}d\xi = \begin{cases} 0, & 若\ l\neq m, \\ \dfrac{2^{1-2\lambda}\pi\Gamma(m+2\lambda)}{[\Gamma(\lambda)]^2(m+\lambda)\Gamma(m+1)}, & 若\ l=m. \end{cases} \qquad (4)$$

这是超球函数的正交性质.

再在 (2) 式中取 $f(\xi)=\xi^l$, 则当 $l\geqslant m$ 时,

$$\int_{-1}^{1}\xi^l P_m^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}d\xi = \binom{l}{m}\frac{2^m\Gamma(m+\lambda)\Gamma(m+2\lambda)}{\Gamma(\lambda)\Gamma(2m+2\lambda)}\int_{-1}^{1}\xi^{l-m}(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi. \qquad (5)$$

当 $l-m$ 是奇数时, 这积分等于 0, 若 $l-m=2k$ 是偶数时, 则由

$$\int_{-1}^{1}\xi^{l-m}(1-\xi^2)^{m+\lambda-\frac{1}{2}}d\xi=\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(m+\lambda+\frac{1}{2}\right)}{\Gamma(k+m+\lambda+1)},$$

得出

$$\int_{-1}^{1}\xi^l P_m^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}d\xi$$
$$=\begin{cases}0, & \text{若 } l<m \text{ 或 } l-m \text{ 的奇数},\\ \dfrac{\pi}{2^{l+2\lambda-1}}\dfrac{l!}{k!(l-2k)!}\dfrac{\Gamma(l-2k+2\lambda)}{\Gamma(\lambda)\Gamma(l-k+\lambda+1)}, & \text{若 } l-m=2k.\end{cases}\tag{6}$$

由于 $P_l^{(\lambda)}(\xi)$ 是一 l 次多项式, 因此任一 m 次多项式可以表成为

$$f(\xi)=\sum_{l=0}^{m}a_l P_l^{(\lambda)}(\xi).\tag{7}$$

乘以 $P_l^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}$, 并由 -1 到 $+1$ 求积分, 得

$$a_l=2^{2\lambda-1}\frac{(\Gamma(\lambda))^2(l+\lambda)\Gamma(l+1)}{x\Gamma(l+2\lambda)}\int_{-1}^{1}f(\xi)P_l^{(\lambda)}(\xi)(1-\xi^2)^{\lambda-\frac{1}{2}}d\xi.\tag{8}$$

由此立即推出

定理 1 一 m 次多项式 $f(\xi)$ 的 $a_l(0\leqslant l<m-1)$ 都等于 0, 则与 $P_m^{(\lambda)}(\xi)$ 相差一常数因子.

联合 (6), (7), (8) 得展开式

$$\xi^m=\frac{m!\Gamma(\lambda)}{2^m}\sum_{0\leqslant k\leqslant\frac{1}{2}m}\frac{m-2k+\lambda}{k!\Gamma(m-k+\lambda+1)}P_{m-2k}^{(\lambda)}(\xi).\tag{9}$$

根据此式今证明恒等式: 当 $v>\lambda>-\dfrac{1}{2}$ 时, 有

$$P_m^{(v)}(\xi)=\frac{\Gamma(\lambda)}{\Gamma(v)}\sum_{0\leqslant k\leqslant\frac{1}{2}m}c_k P_{m-2k}^{(\lambda)}(\xi),\tag{10}$$

此处

$$c_k=\frac{m-2k+\lambda}{k!}\cdot\frac{\Gamma(k+v-\lambda)}{\Gamma(v-\lambda)}\cdot\frac{\Gamma(m+v-k)}{\Gamma(m+\lambda+1-k)}.\tag{11}$$

在证明此式之前, 先叙述有关 Γ 函数的差分公式. 定义 $\Delta f(x)=f(x+1)-f(x)$ 为函数 $f(x)$ 的一级差分, $\Delta^q f(x)=\Delta^{q-1}[\Delta f(x)]$ 是 $f(x)$ 的 q 阶差分, 不难证明

$$\Delta^q f(x)=\sum_{l=0}^{q}(-1)^l\begin{pmatrix}q\\l\end{pmatrix}f(x+q-l).$$

由于

$$\Delta\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}=\frac{\Gamma(\alpha+x+1)}{\Gamma(\beta+x+1)}-\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}=(\alpha-\beta)\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x+1)}.$$

因此, 当 $\alpha>\beta$ 时

$$\Delta^q\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)}=\frac{\Gamma(\alpha-\beta+1)}{\Gamma(\alpha-\beta-q+1)}\frac{\Gamma(\alpha+x)}{\Gamma(\beta+x+q)}. \tag{12}$$

现在来证明 (11) 式. 由 (1.1) 及 (9) 可知

$$\begin{aligned}P_m^{(v)}(\xi)=&\sum_{0\leqslant s\leqslant\frac{1}{2}m}(-1)^s\frac{\Gamma(v+m-s)}{\Gamma(v)\Gamma(s+1)\Gamma(m-2s+1)}(2\xi)^{m-2s}\\=&\frac{\Gamma(\lambda)}{\Gamma(v)}\sum_{0\leqslant s\leqslant\frac{1}{2}m}(-1)^s\frac{\Gamma(v+m-s)}{\Gamma(s+1)}\\&\times\sum_{0\leqslant k\leqslant\frac{m}{2}-s}\frac{m-2s-2k+\lambda}{k!\Gamma(m-2s-k+\lambda+1)}P_{m-2k-2s}^{(\lambda)}(\xi)\\=&\frac{\Gamma(\lambda)}{\Gamma(v)}\sum_{0\leqslant t\leqslant\frac{1}{2}m}c_tP_{m-2t}^{(\lambda)}(\xi),\end{aligned}$$

这里

$$\begin{aligned}c_t=&\sum_{s+k=t}(-1)^s\frac{\Gamma(v+m-s)(m-2s-2k+\lambda)}{s!k!\Gamma(m-2s-k+\lambda+1)}\\=&\frac{m-2t+\lambda}{t!}\sum_{s=0}^{t}(-1)^s\binom{t}{s}\frac{\Gamma(v+m-s)}{\Gamma(m-t-s+\lambda+1)}\\=&\frac{(m-2t+\lambda)\Gamma(t+v-\lambda)\Gamma(v+m-t)}{t!\Gamma(v-\lambda)\Gamma(m-t+\lambda+1)}.\end{aligned}$$

2.3　边界值问题

还是从单位圆谈起: 如果在圆周上有一 Fourier 级数

$$f(e^{i\theta})\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta), \tag{1}$$

$$\begin{cases}a_0=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}f(e^{i\theta})d\theta, a_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}f(e^{i\theta})\cos n\theta d\theta,\\ b_n=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}f(e^{i\theta})\sin n\theta d\theta,\end{cases} \tag{2}$$

则函数

$$f(\rho e^{i\theta})=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\rho^n \tag{3}$$

是以 (1) 为边界值的调和函数, 要看出这点是十分容易的. 首先, 因为 $\rho^n\cos n\theta$, $\rho^n\sin n\theta$ 是以 $\cos n\theta,\sin n\theta$ 为边界值的调和函数, 而 Laplace 方程是线性的. 因此, 可以如此希望, 这也是在某种条件下所证明了的事实.

我们现在的目的是: 把这种想法推到单位球上. 但先指出, 把 (2) 代入 (3) 得

$$\begin{aligned}f(\rho e^{i\theta})=&\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\psi})d\psi+\sum_{n=1}^{\infty}\frac{\rho^n}{\pi}\int_0^{2\pi}f(e^{i\psi})\\&\times(\cos n\psi\cos n\theta+\sin n\psi\sin n\theta)d\psi\\=&\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\psi})d\psi+\sum_{n=1}^{\infty}\frac{\rho^n}{\pi}\int_0^{2\pi}f(e^{i\psi})\cos n(\theta-\psi)d\psi \quad (4)\\=&\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\psi})\left(1+2\sum_{n=1}^{\infty}\rho^n\cos n(\theta-\psi)\right)d\psi\\=&\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\psi})\frac{1-\rho^2}{1-2\rho\cos(\theta-\psi)+\rho^2}d\psi, \quad (5)\end{aligned}$$

这样又得出 Poisson 公式.

从 (4) 建议, 我们能否把球内的任一调和函数 $f(\rho u)$ 表为

$$f(\rho u)=\sum_{n=0}^{\infty}\rho^n\frac{1}{\omega_{n-1}}\int\cdots\int_{vv'=1}f(v)\varPhi_n(u,v)\dot{v} \tag{6}$$

的形式. 如果可以, 则可以希望 $f(\rho u)$ 是以

$$\sum_{n=0}^{\infty}\frac{1}{\omega_{n-1}}\int\cdots\int_{vv'=1}f(v)\varPhi_n(u,v)\dot{v} \tag{7}$$

为边界值的调和函数. 本来应当从单位球上的调和分析出发, 先找出球面上的正交完整系, 然后再研究 (6), 但这条途径较长, 而且还要用到一定的群表示的知识, 我们现在取一条相反的但较方便的途径: 先把 Poisson 核展开然后再涉及其他.

Laplace 方程的 Poisson 核有以下的展开式: 由 (2.1.4), (2.2.10), (2.2.11) 可知: 当 $x=\rho v,vv'=1$ 时,

$$\begin{aligned}\frac{1-xx'}{(1-2xu'+xx')^{n/2}}=&(1-\rho^2)\sum_{m=0}^{\infty}P_m^{\left(\frac{n}{2}\right)}(uv')\rho^m\\=&(1-\rho^2)\frac{\varGamma\left(\frac{n}{2}-1\right)}{\varGamma\left(\frac{n}{2}\right)}\sum_{m=0}^{\infty}\sum_{0\leqslant k\leqslant\frac{1}{2}m}c_kP_{m-2k}^{\left(\frac{n}{2}-1\right)}(uv')\rho^m,\end{aligned}$$

此处 $c_k = m - 2k + \frac{1}{2}n - 1$. 换变数, 命 $l = m - 2k$, 则得

$$\frac{1 - xx'}{(1 - 2xu' + xx')^{n/2}} =(1-\rho^2)\sum_{l=0}^{\infty}\frac{l + \frac{1}{2}n - 1}{\frac{1}{2}n - 1}P_l^{(\frac{1}{2}n-1)}(uv')\rho^l\sum_{k=0}^{\infty}\rho^{2k}$$
$$=\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\rho^l P_l^{(\frac{1}{2}n-1)}(uv').$$

因此 Poisson 积分公式可以写成为

$$\frac{1}{\omega_{n-1}}\int\underset{vv'=1}{\cdots}\int\frac{(1-\rho^2)f(v)}{(1-2\rho\cos uv'+\rho^2)^{\frac{n}{2}}}\dot{v}$$
$$=\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\rho^l\underset{vv'=1}{\int\cdots\int}P_l^{(\frac{n}{2}-1)}(uv')f(v)\dot{v}.$$

由此建议: 在球面上给了一个函数 $f(v)$, 它有展开式

$$f(u) \sim \frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\underset{vv'=1}{\int\cdots\int}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}, \tag{8}$$

则可望有一调和函数

$$f(\rho u) \sim \frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\rho^l\underset{vv'=1}{\int\cdots\int}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}, \tag{9}$$

以 $f(u)$ 为边界值.

展式 (8) 称为 Laplace 级数, 它是 Fourier 级数的自然推广, 超球函数 $P_l^{(\frac{1}{2}n-1)}(\xi)$ 也特别命名为 Legendre 函数或 Legendre 多项式.

关于 (8) 式的收敛求和问题今不深论 (参考陈建功著《直交函数论》). 但可以说明的是如果 (8) 式收敛, 可以保证 (9) 式收敛, 但反之, (9) 式收敛恰不能保证 (8) 式收敛, 甚至不能保证它在边界上定义一个函数.

与 1.7 节证明定理 1 所用的方法相同, 可以证明: 如果 $f(v)$ 是连续函数, 则

$$\lim_{\rho\to 1}\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\rho^l\underset{vv'=1}{\int\cdots\int}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v} = f(u). \tag{10}$$

这里建议, 有 "Abelian" 定理与 "Tauberian" 定理, 也就是如果

$$\lim_{N\to\infty}\frac{1}{\omega_{n-1}}\sum_{l=0}^{N}\frac{2l+n-2}{n-2}\underset{vv'=1}{\int\cdots\int}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v} = s_0, \tag{11}$$

是否有

$$\lim_{\rho\to 1}\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\rho^l\int\cdots\int_{vv'=1}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}=s_0. \tag{12}$$

这就是普通幂级数的 Abel 定理, 当然正确.

反之, 如果 (12) 式成立而且

$$\int\cdots\int_{vv'=1}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}=O\left(\frac{1}{l^2}\right),$$

则 (11) 式也成立.

2.4 球面上的广义函数

现在指出球面上定义广义函数的一个方法. 在边界上 (即球面上) 给了一个连续函数, 我们有一个球内的调和函数以之为边界值, 反之, 一个球内无处不调和的函数的边界函数不一定连续, 甚至于可能不成其为函数. 但我们可以抽象地定义: 球上的一个广义函数可以理解为球内一个调和函数的边界值.

具体地说: 如果有一展开式

$$f(\rho u)=\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\rho^l\int\cdots\int_{vv'=1}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}, \tag{1}$$

而

$$\lim_{l\to\infty}\left|\int\cdots\int_{vv'=1}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}\right|^{-\frac{1}{l}}\leqslant 1,$$

则级数 (1) 在单位球内收敛, 而且定义一调和函数.

形式 Laplace 级数

$$\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\int\cdots\int_{vv'=1}P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}$$

作为一个广义函数的定义.

即以圆而论, 这样定义的广义函数的范围就比 Schwartz 所定义的广义函数的范围为宽, 而且处理起来也比较方便.

2.5 球面上的调和分析

Laplace 级数不是 Fourier 级数的最确切的推广. 因为它并没有根据球面上的完整正交系出发.

命 γ 表示 n 维空间的单位超球面, 即 $u=(u_1,\cdots,u_n)$ 适合于

$$uu'=1 \tag{1}$$

的矢量 u 所成的图形. 超球面 z 有球坐标表示

$$\begin{cases} u_1=\cos\theta_1, \\ u_2=\sin\theta_1\cos\theta_2, \\ \quad\cdots\cdots \\ u_{n-1}=\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ u_n=\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \end{cases} \tag{2}$$

这里

$$0\leqslant\theta_r\leqslant\pi\quad(1\leqslant r\leqslant n-2),\quad 0\leqslant\theta_{n-1}\leqslant 2\pi. \tag{3}$$

超球面的表面积元素是

$$\dot{u}=\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}d\theta_1\cdots d\theta_{n-1}. \tag{4}$$

总面积等于

$$\begin{aligned} \omega=\omega_{n-1}&=\underset{uu'=1}{\int\cdots\int}\dot{u} \\ &=\int_0^{\pi}\sin^{n-2}\theta_1 d\theta_1\cdots\int_0^{\pi}\sin\theta_{n-2}d\theta_{n-2}\int_0^{2\pi}d\theta_{n-1} \\ &=\frac{2\pi^{n/2}}{\Gamma\left(\frac{1}{2}n\right)} \end{aligned} \tag{5}$$

超球面上的调和分析的主要目的是: 找出一组函数

$$\varphi_i(u)=\varphi_i(u_1,\cdots,u_n),\quad i=0,1,2,\cdots$$

在 γ 上正交就范, 即

$$\frac{1}{\omega_{n-1}}\underset{uu'=1}{\int\cdots\int}\varphi_i(u)\varphi_j(u)\dot{u}=\delta_{ij},$$

而且“任一函数” $f(u)$ 可以由 $\varphi_i(u)$ 的线性组合逼近它, 即给了任一 $\varepsilon>0$, 有 $c_0,\cdots,c_n$ 使

$$\left|f(u)-\sum_{i=0}^{M}c_i\varphi_i\right|<\varepsilon,$$

并且定义

$$\sum_{i=0}^{\infty} c_i\varphi_i(u), \quad c_i = \frac{1}{\omega_{n-1}} \int\cdots\int_{vv'=1} f(v)\varphi_i(v)\dot{v}$$

是函数 $f(u)$ 的 Fourier 级数.

具体地给出 $\varphi_i(u)$ 才是 Fourier 级数的最好推广. 但要达到这目的必需要较多的预备知识 (详见《典型域的调和分析》, 第七章, §7.2). 而 Laplace 级数仅是把一个不可约表示的支量不加区别地加在一起的情况而已. 也就是 Laplace 级数中的 $P_m^{(\frac{1}{2}n-1)}(uv')$ 是从

$$\sum_{i=0}^{\infty} c_i\varphi_i(u) = \frac{1}{\omega_{n-1}} \sum_{i=0}^{\infty} \int\cdots\int_{vv'=1} f(v)\varphi_i(u)\varphi_i(v)\dot{v}$$

中若干个 $\varphi_i(u)\varphi_i(v)$ 加在一起而得来的.

2.6 不变方程的 Poisson 核的展开

再看

$$\left(\frac{1-xx'}{1-2xu'+xx'}\right)^{n-1}$$

的展开式. 命 $x=\rho v, vv'=1$, 由 (2.1.4), (2.2.10), (2.2.11) 推得

$$\begin{aligned}\left(\frac{1-xx'}{1-2xu'+xx'}\right)^{n-1} &= \left(\frac{1-\rho^2}{1-2\rho uv'+\rho^2}\right)^{n-1}\\ &=(1-\rho^2)^{n-1}\sum_{m=0}^{\infty} P_m^{(n-1)}(uv')\rho^m\\ &=(1-\rho^2)^{n-1}\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}n-1\right)}{\Gamma(n-1)}\sum_{0\leqslant k\leqslant\frac{1}{2}m} c_k P_{m-2k}^{(\frac{1}{2}n-1)}(uv')\rho^m\\ &=(1-\rho^2)^{n-1}\frac{\Gamma\left(\frac{1}{2}n-1\right)}{\Gamma(n-1)}\sum_{l=0}^{\infty}\psi_l(\rho)P_l^{(\frac{1}{2}n-1)}(uv'), \qquad (1)\end{aligned}$$

这里

$$\begin{aligned}\psi_l(\rho) &=\rho^l\sum_{k=0}^{\infty} c_k\rho^{2k}\\ &=\rho^l\sum_{k=0}^{\infty}\frac{l+\frac{1}{2}n-1}{k!}\frac{\Gamma\left(k+\frac{1}{2}n\right)}{\Gamma\left(\frac{1}{2}n\right)}\frac{\Gamma(l+k+n-1)}{\Gamma\left(l+k+\frac{1}{2}n\right)}\rho^{2k}\end{aligned}$$

$$=\rho^l\frac{\Gamma(l+n-1)}{\Gamma\left(l+\frac{1}{2}n-1\right)}F\left(\frac{1}{2}n,l+n-1;l+\frac{1}{2}n;\rho^2\right),$$

其中 $F(\alpha,\beta;\gamma;x)$ 是超几何级数.

由超几何级数的性质:

$$F(\alpha,\beta;\gamma;x)=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;x)$$

及当 $\alpha+\beta-\gamma<0$ 时,

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

可得

$$\begin{aligned}&\left(\frac{1-\rho^2}{1-2\rho uv'+\rho^2}\right)^{n-1}\\&=\frac{\Gamma\left(\frac{1}{2}n-1\right)}{\Gamma(n-1)}\sum_{l=0}^{\infty}\frac{\Gamma(l+n-1)}{\Gamma\left(l+\frac{1}{2}n-1\right)}\rho^lF\left(l,-\frac{1}{2}n+1;l+\frac{1}{2}n;\rho^2\right)P_l^{\left(\frac{1}{2}n-1\right)}(uv')\\&=\frac{\Gamma\left(\frac{1}{2}n-1\right)}{\Gamma(n-1)}\sum_{l=0}^{\infty}\frac{\Gamma(l+n-1)\Gamma(n-1)\Gamma\left(l+\frac{1}{2}n\right)}{\Gamma\left(l+\frac{1}{2}n-1\right)\Gamma\left(\frac{1}{2}n\right)\Gamma(l+n-1)}\tau_l(\rho)\rho^lP_l^{\left(\frac{1}{2}n-1\right)}(uv')\\&=\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\tau_l(\rho)\rho^lP_l^{\left(\frac{1}{2}n-1\right)}(uv'),\end{aligned}\tag{2}$$

这里

$$\tau_l(\rho)=\frac{\Gamma\left(\frac{1}{2}n\right)\Gamma(l+n-1)}{\Gamma(n-1)\Gamma\left(l+\frac{1}{2}n\right)}\times F\left(l,-\frac{1}{2}n+1;l+\frac{1}{2}n;\rho^2\right)\tag{3}$$

适合于

$$\lim_{\rho\to1}\tau_l(\rho)=1.\tag{4}$$

因此, Poisson 公式可以写成为

$$\begin{aligned}&\frac{1}{\omega_{n-1}}\int\cdots\int_{vv'=1}\left(\frac{1-\rho^2}{1-2\rho uv'+\rho^2}\right)^{n-1}f(v)\dot{v}\\&=\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\tau_l(\rho)\rho^l\int\cdots\int_{vv'=1}P_l^{\left(\frac{1}{2}n-1\right)}(uv')f(v)\dot{v}.\end{aligned}\tag{5}$$

也就是如果给了一个函数 $f(v)$ 具有收敛的 Laplace 级数

$$\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\int\cdots\int_{vv'=1} P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v}, \tag{6}$$

则在球内不变方程有解

$$\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l+n-2}{n-2}\tau_l(\rho)\cdot\rho^l\int\cdots\int_{vv'=1} P_l^{(\frac{1}{2}n-1)}(uv')f(v)\dot{v},$$

它以 $f(v)$ 为其边界值.

附记 现在的处理方法是希望保持旧有的球面上的调和分析, 不惜把 ρ^l 改为 $\rho^l\tau_l(\rho)$. 另一方面, 可以保持因子 ρ^l 而改变所对应的 Legendre 函数. 例如把 (1) 改为

$$(1-\rho^2)^{n-1}\sum_{m=0}^{\infty}P_m^{(n-1)}(uv')\rho^m=\sum_{l=0}^{\infty}Q_l(uv')\rho^l,$$

这里

$$Q_l(\xi)=\sum_{0\leqslant k\leqslant\min(n-1,\frac{1}{2}l)}(-1)^k\binom{n-1}{k}P_{l-2k}^{(n-1)}(\xi).$$

然后再继续做下去.

2.7 完 备 性

前已证明, 球上任一连续函数 $\varphi(u)$ 常有

$$\varphi(u)=\lim_{\rho\to1}\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l-n+2}{n-2}\rho^l\int\cdots\int_{vv'=1} P_l^{(\frac{1}{2}n-1)}(uv')\varphi(v)\dot{v}.$$

由此推得: 给予任一 $\varepsilon>0$, 我们有 ρ 使

$$\left|\varphi(u)-\frac{1}{\omega_{n-1}}\sum_{l=0}^{\infty}\frac{2l-n+2}{n-2}\rho^l\int\cdots\int_{vv'=1} P_l^{(\frac{1}{2}n-1)}(uv')\varphi(v)\dot{v}\right|<\frac{1}{2}\varepsilon,$$

又有 N 使

$$\left|\varphi(u)-\frac{1}{\omega_{n-1}}\sum_{l=0}^{N}\frac{2l-n+2}{n-2}\rho^l\int\cdots\int_{vv'=1} P_l^{(\frac{1}{2}n-1)}(uv')\varphi(v)\dot{v}\right|<\varepsilon. \tag{1}$$

我们现在证明: 如果球上一个连续函数适合于

$$\int\cdots\int_{uu'=1}\varphi(u)P_l^{(\frac{1}{2}n-1)}(uv')\dot{u}=0,\quad l=0,1,2,\cdots, \tag{2}$$

则 $\varphi(u)\equiv 0$.

这是极易证明的事实, 由 (1) 可知

$$\frac{1}{\omega_{n-1}}\underset{uu'=1}{\int\cdots\int}\left(\varphi(u)-\frac{1}{\omega_{n-1}}\sum_{l=0}^{N}\frac{2l-n+2}{n-2}\rho^l\right.\\ \left.\times\underset{vv'=1}{\int\cdots\int}P_l^{(\frac{1}{2}n-1)}(uv')\varphi(v)\dot{v}\right)^2\dot{u}<\varepsilon. \tag{3}$$

乘开积分, 由 (2) 可知左边等于

$$\frac{1}{\omega_{n-1}}\underset{uu'=1}{\int\cdots\int}[\varphi(u)]^2\dot{u}\\ +\frac{1}{\omega_{n-1}}\underset{uu'=1}{\int\cdots\int}\left(\frac{1}{\omega_{n-1}}\sum_{l=0}^{N}\frac{2l-n+2}{n-2}\rho^l\underset{vv'=1}{\int\cdots\int}P_l^{(\frac{1}{2}n-1)}(uv')\varphi(v)\dot{v}\right)^2\dot{u}\\ \geqslant\frac{1}{\omega_{n-1}}\underset{uu'=1}{\int\cdots\int}[\varphi(u)]^2\dot{u}$$

由 (3) 推出: 给任一 $\varepsilon>0$, 常有

$$\frac{1}{\omega_{n-1}}\underset{uu'=1}{\int\cdots\int}[\varphi(u)]^2\dot{u}<\varepsilon.$$

除 $\varphi(u)\equiv 0$ 外, 这是不可能的.

2.8　解偏微分方程 $\partial_u^2\Phi=\lambda\Phi$

考虑球面上的偏微分方程

$$\partial_u^2\Phi=\lambda\Phi. \tag{1}$$

使此式有解的 λ 称为特征值. 今往证

定理 1　除 $\lambda=-l(l+n-2)(l=0,1,2,\cdots)$ 外无其他的特征值, 而 $P_l^{(\frac{1}{2}n-1)}(uv')$ 是对应此特征值的解.

$$\partial_u^2P_l^{(\frac{1}{2}n-1)}(uv')=-l(l+n-2)P_l^{(\frac{1}{2}n-1)}(uv'). \tag{2}$$

取 $v=(1,0,\cdots,0)$, 则 (2) 简化为

$$\left(\frac{\partial^2}{\partial\theta_1^2}+(n-2)\cot\theta_1\frac{\partial}{\partial\theta_1}\right)P_l^{(\frac{1}{2}n-1)}(\cos\theta_1)\\ =-l(l+n-2)P_l^{(\frac{1}{2}n-1)}(\cos\theta_1), \tag{3}$$

换变数, 命 $\cos\theta_1=\xi$, 则 (3) 等价于

$$\left[(1-\xi^2)\frac{\partial^2}{\partial\xi^2}-(n-1)\xi\frac{\partial}{\partial\xi}\right]P_l^{(\frac{1}{2}n-1)}(\xi)=-l(l+n-2)P_l^{(\frac{1}{2}n-1)}(\xi),$$

这就是公式 (1.6), 因此当 $v=(1,0,\cdots,0)$ 时, (2) 式是正确的.

假定 v 是任一单位矢量, 有正交方阵 Γ 使 $v\Gamma=(1,0,\cdots,0)$, 引进球坐标

$$u\Gamma=(\cos\theta_1,\sin\theta_1\cos\theta_2,\cdots),$$

则问题化为与上面所讲的完全一样.

再证如果

$$\partial_u^2\Phi=\lambda\Phi,\quad \partial_u^2\Psi=\mu\Psi,\quad \lambda\neq\mu, \tag{4}$$

则

$$\underset{vv'=1}{\int\cdots\int}\Phi(v)\Psi(v)\dot{v}=0. \tag{5}$$

由于

$$\underset{vv'=1}{\int\cdots\int}\Phi(v)\partial_v^2\Psi(v)\dot{v}=\sum_{p=1}^{n-1}J_p,$$

此处

$$J_p=\underset{vv'=1}{\int\cdots\int}\Phi(v)\frac{\partial}{\partial\theta_p}\left(\sin^{n-p-1}\theta_p\frac{\partial\Psi}{\partial\theta_p}\right)\times\frac{\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}d\theta_1\cdots d\theta_{n-1}}{\sin^2\theta_1\cdots\sin^2\theta_{p-1}\sin^{n-p-1}\theta_p}.$$

当 $p<n-1$ 时, J_p 中有一单积分

$$\begin{aligned}&\int_0^\pi\Phi\frac{\partial}{\partial\theta_p}\left(\sin^{n-p-1}\theta_p\frac{\partial\Psi}{\partial\theta_p}\right)d\theta_p\\=&\Phi\sin^{n-p-1}\theta_p\left|\frac{\partial\Psi}{\partial\theta_p}\right|_0^\pi-\int_0^\pi\frac{\partial\Phi}{\partial\theta_p}\sin^{n-p-1}\theta_p\frac{\partial\Psi}{\partial\theta_p}d\theta_p\\=&\int_0^\pi\Psi\frac{\partial}{\partial\theta_p}\left(\sin^{n-p-1}\theta_p\frac{\partial\Phi}{\partial\theta_p}\right)d\theta_p,\end{aligned}$$

即 Φ 与 Ψ 易地而处. 当 $p=n-1$ 时, 方法也相仿, 但需注意函数 Φ,Ψ 对 θ_{n-1} 的周期性. 因此得到

$$\underset{vv'=1}{\int\cdots\int}\Phi(v)\partial_v^2\Psi(v)\dot{v}=\underset{vv'=1}{\int\cdots\int}\Psi(v)\partial_v^2\Phi(v)\dot{v}. \tag{6}$$

由此及 (4) 可知

$$(\lambda-\mu)\underset{vv'=1}{\int\cdots\int}\Phi(v)\Psi(v)\dot{v}=0,$$

即得 (5) 式.

现在证明除

$$\lambda = -l(l+n-2)$$

外, (1) 式无其他的特征根. 同时也将证明, 当 $\lambda = -l(l+n-2)$ 时, 除 $P_l^{(\frac{1}{2}n-1)}(uv')$ 外无其他的特征函数. 注意, 这里并不仅仅是一个特征函数, 给予任一球面上的一点 v, 有一特征函数 $P_l^{(\frac{1}{2}n-1)}(uv')$, 这些特征函数成一线性空间 (这空间的维数等于

$$\binom{n+l-1}{l} - \binom{n+l-3}{l-2},$$

现在不去证明).

如果 λ 是另一特征根, 而 $\Phi(v)$ 是对应的特征函数, 则

$$\int \underset{vv'=1}{\cdots} \int \Phi(v) P_l^{(\frac{1}{2}n-1)}(uv')\dot{v} = 0, \quad l = 0, 1, 2, \cdots.$$

由完备性可知 $\Phi(v) = 0$. 由完备性易知, 当 $\lambda = -l(l+n-2)$ 时, $P_l^{(\frac{1}{2}n-1)}(uv')$ 表示了对应于此特征根的所有的特征函数.

2.9　附　　记

我们也可以从 Poisson 核的性质推出超球函数的整套结果来. 例如:

(1) 首先

$$H(x,y) = \frac{1 - xx'yy'}{(1 - 2xy' + xx'yy')^{n/2}}, \quad xx' < 1, yy' < 1 \tag{1}$$

是 $(x_1, \cdots, x_n)$ 的调和函数. 直接证明之如次:

$$\frac{\partial}{\partial x_i} H(x,y) = \frac{-2x_i yy'}{(1 - 2xy' + xx'yy')^{n/2}} + \frac{n(1 - xx'yy')(y_i - x_i yy')}{(1 - 2xy' + xx'yy')^{\frac{n}{2}+1}}$$

及

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2} H(x,y) = &- \frac{2yy'}{(1 - 2xy' + xx'yy')^{n/2}} \\ &- \frac{4nx_i yy'(y_i - x_i yy')}{(1 - 2xy' + xx'yy')^{\frac{n}{2}+1}} - \frac{n(1 - xx'yy')yy'}{(1 - 2xy' + xx'yy')^{\frac{n}{2}}} \\ &+ n(n+2)\frac{(1 - xx'yy')(y_i - x_i yy')^2}{(1 - 2xy' + xx'yy')^{\frac{n}{2}+2}}.\end{aligned}$$

因此

$$\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}H(x,y)=-\frac{2nyy'}{(1-2xy'+xx'yy')^{\frac{n}{2}}}$$
$$-\frac{4nyy'(xy'-xx'yy')}{(1-2xy'+xx'yy')^{\frac{n}{2}+1}}-\frac{n^2(1-xx'yy')yy'}{(1-2xy'+xx'yy')^{\frac{n}{2}+1}}$$
$$+\frac{n(n+2)(1-xx'yy')[yy'-2xy'yy'+xx'(yy')^2]}{(1-2xy'+xx'yy')^{\frac{n}{2}+2}}$$
$$=0.$$

(2) 我们有等式

$$H(x,y)=\frac{1}{\omega_{n-1}}\int\cdots\int_{vv'=1}H(x,v)H(v,y)\dot{v},$$
$$xx'<1,\quad yy'<1. \tag{2}$$

这结果的证明是十分简单的, 其原因是 $H(x,v)$ 就是 Poisson 核, 而 $H(v,y)$ 是调和函数 $H(x,y)$ 的边界值.

(3) 命 $x=\rho u, y=rv$, 命

$$H(\rho u,rv)=\frac{1-\rho^2r^2}{(1-2\rho ruv'+\rho^2r^2)^{n/2}}=\sum_{l=0}^{\infty}Q_l(uv')(\rho r)^l \tag{3}$$

是幂级数展开式, 代入 (2) 中得

$$\sum_{l=0}^{\infty}Q_l(uv')(\rho r)^l=\frac{1}{\omega_{n-1}}\int\cdots\int_{ww'=1}\sum_{p=0}^{\infty}Q_p(uw')\rho^p\sum_{q=0}^{\infty}Q_q(wv')r^q\dot{w},$$

逐项求积分并比较系数得

$$\frac{1}{\omega_{n-1}}\int\cdots\int_{ww'=1}Q_p(uw')Q_q(wv')\dot{w}=\begin{cases}0, & 若\ p\neq q,\\ Q_p(uv'), & 若\ p=q.\end{cases} \tag{4}$$

(4) 取特例 $u=v=(1,0,\cdots,0), w=(\cos\theta_1,\sin\theta_1\cos\theta_2,\cdots), \dot{w}=\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}d\theta_1\cdots d\theta_{n-1}$, 得

$$\frac{1}{\omega_{n-1}}\int_0^{\pi}Q_p(\cos\theta_1)Q_q(\cos\theta_1)\sin^{n-2}\theta_1 d\theta$$
$$\times\int_0^{\pi}\sin^{n-3}\theta_2 d\theta_2\cdots\int_0^{\pi}\sin\theta_{n-2}d\theta_{n-2}\int_0^{2\pi}d\theta_{n-1}$$
$$=\begin{cases}0, & 若\ p\neq q,\\ Q_p(1), & 若\ p=q.\end{cases}$$

由于

$$\omega_{n-1}=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)},\quad \int_0^\pi \sin^p\theta d\theta=\sqrt{\pi}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}$$

及 $Q_p(1)$ 是 $(1+x)(1-x)^{-(n-1)}$ 的展式中 x^p 的系数, 即

$$\begin{aligned}Q_p(1)=&\frac{(n-1)n\cdots(n+p-3)(n+2p-2)}{p!}\\=&\frac{(n+p-3)!(n+2p-2)}{(n-2)!p!},\end{aligned}$$

因此得出

$$\begin{aligned}&\int_0^\pi Q_p(\cos\theta_1)Q_q(\cos\theta_1)\sin^{n-2}\theta_1 d\theta_1\\=&\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\frac{(n+p-3)!}{(n-2)!p!}(n+2p-2)\delta_{pq}.\end{aligned}$$

命 $\cos\theta_1=\xi$, 即得

$$\begin{aligned}&\int_{-1}^1 Q_p(\xi)Q_q(\xi)(1-\xi^2)^{\frac{1}{2}(n-3)}d\xi\\=&\frac{(n+2p-2)2^{2-n}\pi\cdot(n+p-3)!}{\Gamma\left(\frac{n}{2}\right)^2 p!}\delta_{pq}.\end{aligned}\tag{5}$$

(5) 把幂级数

$$H(\rho u,v)=\sum_{l=0}^{\infty}Q_l(uv')\rho^l$$

代入 Laplace 方程

$$\frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}(H(\rho u,v))\right)+\frac{1}{\rho^2}\partial_u^2 H(\rho u,v)=0,$$

并比较 ρ^{l-2} 的系数得

$$l(n+l-2)Q_l(uv')+\partial_u^2 Q_l(uv')=0.$$

取特例 $v=(1,0,\cdots,0)$ 及 $\cos\theta=\xi$, 即得: $Q_l(\xi)$ 适合于二阶微分方程:

$$(1-\xi^2)\frac{d^2Q_l}{d\xi^2}-(n-1)\xi\frac{dQ_l}{d\xi}+l(l+n-2)Q_l=0.\tag{6}$$

(6) 如前证明 $Q_l(uv')$ 在球上的完整性, 及 Q_l 在 $(-1,+1)$ 中的完整性等等.

总之, 重用 Poisson 公式可以推出超球函数的重要性质, 这样的推导方法, 把超球函数的研究和 Laplace 方程的研究更打成一片了.

习题 试证

$$\int_0^\pi Q_m(\xi\xi' + \sqrt{1-\xi^2}\sqrt{1-\xi'^2}\cos\theta)\sin^{n-3}\theta d\theta = cQ_m(\xi)Q_m(\xi'),$$

并定出 c 来.

第 3 讲　扩充空间与球几何

3.1　二次变形与扩充空间

以上我们把单位圆推广到单位球, 现在我们把整个平面 (Gauss 平面连 ∞ 点) 及在这平面上起作用的 Möbius 群都推广到 n 维空间. 我们现在讲得抽象些, 但读者可以与第 1 讲相仿, 或通过使单位球不变的变形的形式而思考这些群是怎样想出来的.

我们从一个二次型

$$x_1^2+\cdots+x_n^2-y_1y_2=0 \tag{1}$$

出发, 它的方阵是 $n+2$ 行列的方阵

$$J=\begin{pmatrix} I^{(n)} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \tag{2}$$

我们考虑射影变换

$$(\xi^*,\eta_1^*,\eta_2^*)=\rho(\xi,\eta_1,\eta_2)M, \tag{3}$$

这里 ξ,ξ^* 是 n 维实矢量, $\eta_1,\eta_2,\eta_1^*,\eta_2^*$ 是实数, M 是

$$MJM'=J. \tag{4}$$

显然, 变换 (3) 保持关系

$$\xi\xi'=\eta_1\eta_2. \tag{5}$$

命

$$M=\begin{pmatrix} T & u_1' & u_2' \\ v_1 & a & b \\ v_2 & c & d \end{pmatrix} \tag{6}$$

则得

$$\begin{cases} \xi^*=\rho(\xi T+\eta_1v_1+\eta_2v_2), \\ \eta_1^*=\rho(\xi u_1'+\eta_1a+\eta_2c), \\ \eta_2^*=\rho(\xi u_2'+\eta_1b+\eta_2d). \end{cases} \tag{7}$$

研究非齐次坐标：命 $x=\xi/\eta_2$(是一矢量), $y=\xi^*/\eta_2^*$, 由关系 (5) 可知

$$\frac{\eta_1}{\eta_2}=xx',\quad \frac{\eta_1^*}{\eta_2^*}=yy'.$$

(7) 变成为变形

$$\begin{cases} y=\dfrac{xT+xx'v_1+v_2}{xu_2'+xx'b+d}, \\ yy'=\dfrac{xu_1'+xx'a+c}{xu_2'+xx'b+d}. \end{cases} \tag{8}$$

注意 (8) 的第二式可由第一式推出.

由

$$M^{-1}=JM'J^{-1}=\begin{pmatrix} T' & -2v_2' & -2v_1' \\ -\dfrac{1}{2}u_2 & d & b \\ -\dfrac{1}{2}u_1 & c & a \end{pmatrix}.$$

可以算得 (8) 的逆变换是

$$\begin{cases} x=\dfrac{yT'-\dfrac{1}{2}yy'u_2-\dfrac{1}{2}u_1}{-2yv_1'+yy'b+a}, \\ xx'=\dfrac{-2yv_2'+yy'd+c}{-2yv_1'+yy'b+a}. \end{cases} \tag{9}$$

所有的形如 (8) 的变换成一群, 以 G 表之.

对应于 $\eta_2=0$ 的点称为无穷远点, 添进这无穷远点, 所得出的空间称为扩充空间. 当 $\eta_2=0$, 则 $\xi\xi'=0$, 因之 $\xi=0$, 而 $(\xi,\eta_1,\eta_2)=\eta_1(0,1,0)$ 唯一决定.

注意：由

$$xu_2'+xx'b+d=0 \tag{10}$$

仅得一点. 首先由 M^{-1} 的性质, 可知

$$-\frac{1}{4}u_2u_2'+bd=0.$$

如果 $b=0$, 则 $u_2=0$, (10) 式无解, 即 (8) 把 ∞ 变为 ∞. 如果 $b\neq 0$, 则 (10) 式可以改写成为

$$\left(x+\frac{u_2}{2b}\right)\left(x+\frac{u_2}{2b}\right)'=0,$$

即 (8) 把 $x=-\dfrac{u_2}{2b}$ 变为 ∞.

问题：把扩充空间变为自己, 双有理变形是否就是 (8) 的形式.

不难证明扩充空间成一可递集, 任意二点可以同时变为 $0, \infty$; 使 $0, \infty$ 不变的变形是

$$y = \frac{1}{d} xT,$$

此处 $TT' = I, a = \frac{1}{d}$. 因此, 任一矢量 $x = \rho u$ 可以变为 e, 取 $d = \rho$ 及 $uT = e$, 即任意三点可以变为任意三点. 任意四点的不变量是什么?

3.2 微分度量, 共形映照

由

$$(y, yy', 1) = \rho(x, xx', 1)M, \tag{1}$$

求微分得

$$(dy, 2ydy', 0) = [d\rho(x, xx', 1) + \rho(dx, 2xdx', 0)]M. \tag{2}$$

由于

$$(dy, 2ydy', 0)J(dy, 2ydy', 0)' = dydy'$$

及

$$(x, xx', 1)J(x, xx', 1)' = 0,$$

$$(x, xx', 1)J(dx, 2xdx', 0)' = 0,$$

所以

$$dydy' = \rho^2 dxdx'. \tag{3}$$

由于

$$1 = \rho(xu_2' + xx'b + d),$$

因此

$$dydy' = \frac{dxdx'}{(xu_2' + xx'b + d)^2}. \tag{4}$$

这说明了, 这变形把无穷小球变为无穷小球, 并且是保角变换.

由逆变换得

$$dxdx' = \frac{dydy'}{(-2yv_1' + yy'b + a)^2}. \tag{5}$$

由此二式建议

$$(xu_2' + xx'b + d)(-2yv_1' + yy'b + a) = 1. \tag{6}$$

(6) 式的直接证明是容易的, 因为由 (1) 可知

$$1=\rho(xu_2'+xx'b+d),$$

又由 $(y,yy',1)M^{-1}=\rho(x,xx',1)$ 可知

$$\rho=-2yv_1'+yy'b+a.$$

现在来证明

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 u}{\partial y_i^2}=&\lambda^n\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\lambda^{2-n}\frac{\partial u}{\partial x_i}\right)\\=&\lambda^2\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+(2-n)\lambda\sum_{i=1}^{n}\frac{\partial\lambda}{\partial x_i}\frac{\partial u}{\partial x_i},\end{aligned}\tag{7}$$

这里 $\lambda=xu_2'+xx'b+d$.

由 (4) 可知

$$\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_s}\frac{\partial y_i}{\partial x_t}=\frac{1}{\lambda^2}\delta_{st},\tag{8}$$

乘以 $\dfrac{\partial x_t}{\partial y_j}$, 而对 t 相加得

$$\frac{\partial y_j}{\partial x_s}=\frac{1}{\lambda^2}\frac{\partial x_s}{\partial y_j}.\tag{9}$$

因此

$$\begin{aligned}\frac{\partial u}{\partial y_i}=&\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial y_i},\\ \sum_{i=1}^{n}\frac{\partial^2 u}{\partial y_i^2}=&\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial x_j\partial x_k}\sum_{i=1}^{n}\frac{\partial x_j}{\partial y_i}\frac{\partial x_k}{\partial y_i}+\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\sum_{i=1}^{n}\frac{\partial^2 x_j}{\partial y_i^2}\\=&\lambda^4\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial x_j\partial x_k}\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_j}\cdot\frac{\partial y_i}{\partial x_k}+\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\sum_{i=1}^{n}\frac{\partial^2 x_j}{\partial y_i^2}\\=&\lambda^2\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\sum_{i=1}^{n}\frac{\partial^2 x_j}{\partial y_i^2}.\end{aligned}$$

与 (7) 相比较, 我们看出所待证者是

$$\sum_{i=1}^{n}\frac{\partial^2 x_j}{\partial y_i^2}=(2-n)\lambda\frac{\partial\lambda}{\partial x_j}.\tag{10}$$

由 (8) 及 (9) 可知

$$\sum_{i=1}^{n}\frac{\partial x_i}{\partial y_p}\frac{\partial x_i}{\partial y_q}=\lambda^2\delta_{pq}.$$

微分之得

$$\sum_{i=1}^{n}\frac{\partial^2 x_i}{\partial y_p^2}\frac{\partial x_i}{\partial y_q}+\sum_{i=1}^{n}\frac{\partial x_i}{\partial y_p}\frac{\partial^2 x_i}{\partial y_p \partial y_q}=\frac{\partial \lambda^2}{\partial y_p}\delta_{pq},$$

即

$$\sum_{i=1}^{n}\frac{\partial^2 x_i}{\partial y_p^2}\frac{\partial x_i}{\partial y_q}+\frac{1}{2}\frac{\partial}{\partial y_q}\left(\sum_{i=1}^{n}\left(\frac{\partial x_i}{\partial y_p}\right)^2\right)=\frac{\partial \lambda^2}{\partial y_p}\delta_{pq},$$

即

$$\sum_{i=1}^{n}\frac{\partial^2 x_i}{\partial y_p^2}\frac{\partial x_i}{\partial y_q}=\frac{\partial \lambda^2}{\partial y_p}\delta_{pq}-\frac{1}{2}\frac{\partial \lambda^2}{\partial y_q}.$$

乘以 $\dfrac{\partial y_q}{\partial x_r}$ 而对 q 相加, 得到

$$\frac{\partial^2 x_r}{\partial y_p^2}=\frac{\partial \lambda^2}{\partial y_p}\frac{\partial y_p}{\partial x_r}-\frac{1}{2}\frac{\partial \lambda^2}{\partial x_r},$$

对 p 相加, 即得 (10) 式, 因而得出 (7) 式.

更深入些, 我们利用恒等式

$$\frac{\partial}{\partial x_i}\left(r^2\frac{\partial}{\partial x_i}(r^{-1}\Phi)\right)=r\frac{\partial^2\Phi}{\partial x_i^2}-\Phi\frac{\partial^2 r}{\partial x_i^2}. \tag{11}$$

由 (7) 可以推出: 命

$$Y(y)=X(x)\lambda^{\frac{n}{2}-1},$$

则

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2}&=\lambda^n\sum_{i=1}^{n}\frac{\partial}{\partial x_t}\left((\lambda^{1-\frac{n}{2}})^2\frac{\partial(\lambda^{\frac{n}{2}-1}X)}{\partial x_i}\right)\\&=\lambda^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}-\lambda^n X\sum_{i=1}^{n}\frac{\partial^2\lambda^{\frac{n}{2}-1}}{\partial x_i^2}.\end{aligned}$$

不难直接证明 $\lambda^{-\frac{n}{2}+1}$ 是调和函数, 因而第二项等于 0, 因此得出

$$\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2}=\lambda^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}. \tag{12}$$

3.3　球 变 为 球

球可以写成为

$$\eta_1 yy'+y\xi'+\eta_2=0. \tag{1}$$

它可以用一个 $n+2$ 维的矢量 (ξ,η_1,η_2) 表之.

经变形 (1.8), 球 (1) 变为

$$\eta_1(xu_1'+xx'a+c)+(xT+xx'v_1+v_2)\xi'+\eta_2(xu_2'+xx'b+d)=0,$$

这也是一球, 这球以 $(\xi^*,\eta_1^*,\eta_2^*)$ 表之, 则得

$$\begin{pmatrix}\xi^{*\prime}\\ \eta_1^*\\ \eta_2^*\end{pmatrix}=\begin{pmatrix}T & u_1' & u_2'\\ v_1 & a & b\\ v_2 & c & d\end{pmatrix}\begin{pmatrix}\xi'\\ \eta_1\\ \eta_2\end{pmatrix},$$

即得

$$(\xi^*,\eta_1^*,\eta_2^*)=(\xi,\eta_1,\eta_2)M'. \tag{2}$$

凑方 (1) 可以改写成为

$$\left(y-\frac{\xi}{2\eta_1}\right)\left(y-\frac{\xi}{2\eta_1}\right)'=\frac{\xi\xi'}{4\eta_1^2}-\frac{\eta_2}{\eta_1}=\frac{\xi\xi'-4\eta_1\eta_2}{4\eta_1^2}.$$

因此, 视

$$(\xi,\eta_1,\eta_2)J^{-1}(\xi,\eta_1,\eta_2)'=\xi\xi'-4\eta_1\eta_2\gtreqless 0$$

而球 (1) 表实球、点球或虚球.

不难证明变形 (1.8) 可以把实球、点球与虚球各变为

$$xx'=1,\quad xx'=0,\quad xx'=-1.$$

实球成一可递集, 注意平面也是实球, 特别 $x_1=0$ 就是一实球.

把上半空间 $x_1>0$ 变为单位球 $yy'<1$ 内部的变换是

$$y=\frac{(xx'-1,2x_2,\cdots,2x_n)}{(x_1+1)^2+x_2^2+\cdots+x_n^2}.$$

要证明这点是容易的, 因为

$$yy'=\frac{(xx'-1)^2+4(xx'-x_1^2)}{(1+2x_1+xx')^2}=\frac{(1+xx')^2-4x_1^2}{(1+xx'+2x_1)^2}=\frac{1+xx'-2x_1}{1+xx'+2x_1}$$

及

$$1-yy'=\frac{4x_1}{1+xx'+2x_1},$$

所以把 $x_1>0$ 变为 $yy'<1$. 它把平面 $\xi_1=0$ 变为单位球面 $vv'=1$, 即

$$v=\frac{(\xi_2^2+\cdots+\xi_n^2-1,2\xi_2,\cdots,2\xi_n)}{1+\xi_2^2+\cdots+\xi_n^2}.$$

因此

$$\begin{aligned}
&1-2yv'+yy'\\
=&2\frac{1+xx'}{1+xx'+2x_1}-2\frac{(1-xx')(1-\xi_2^2-\cdots-\xi_n^2)+4(x_2\xi_2+\cdots+x_n\xi_n)}{(1+xx'+2x_1)(1+\xi_2^2+\cdots+\xi_n^2)}\\
=&4\frac{xx'+\xi_2^2+\cdots+\xi_n^2+2(x_2\xi_2+\cdots+x_n\xi_n)}{(1+xx'+2x_1)(1+\xi_2^2+\cdots+\xi_n^2)}\\
=&4\frac{x_1^2+(\xi_2+x_2)^2+\cdots+(\xi_n+x_n)^2}{(1+xx'+2x_1)(1+\xi_2^2+\cdots+\xi_n^2)}.
\end{aligned}$$

所以

$$\begin{aligned}
\left(\frac{1-yy'}{1-2y\dot{v}'+yy'}\right)^{n-1}\dot{v}&=\left(\frac{x_1(1+\xi_2^2+\cdots+\xi_n^2)}{x_1^2+(\xi_2+x_2)^2+\cdots+(\xi_n+x_n)^2}\right)^{n-1}\\
&\quad\times\frac{d\xi_2\cdots d\xi_n}{\left[\frac{1}{2}(1+\xi_2^2+\cdots+\xi_n^2)\right]^{n-1}}\\
&=\left(\frac{2x_1}{x_1^2+(\xi_2+x_2)^2+\cdots+(\xi_n+x_n)^2}\right)^{n-1}d\xi_2\cdots d\xi_n.
\end{aligned}$$

即得“上半”空间的 Poisson 公式

$$\Phi(x_1,\cdots,x_n)=\frac{1}{\omega_{n-1}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{(2x_1)^{n-1}\Phi(0,\xi_2,\cdots,\xi_n)d\xi_2\cdots d\xi_n}{[x_1^2+(\xi_2+x_2)^2+\cdots+(\xi_n+x_n)^2]^{n-1}},$$

它所适合的偏微分方程是

$$x_1^2\sum_{i=1}^{n}\frac{\partial^2\Phi}{\partial x_i^2}+(2-n)x_1\frac{\partial\Phi}{\partial x_1}=0. \tag{3}$$

又从 Laplace 方程关于单位球的 Poisson 公式

$$\Phi(y)=\frac{1}{\omega_{n-1}}\int\cdots\int_{vv'=1}\frac{1-yy'}{(1-2yv'+yy')^{n/2}}\Phi(v)\dot{v}, \tag{4}$$

行变化

$$\Phi(y)=\Psi(x)\lambda^{\frac{n}{2}-1}=\Psi(x)(1+xx'+2x_1)^{\frac{n}{2}-1}, \tag{5}$$

则

$$\Phi(v)=\Phi(0,\xi_2,\cdots,\xi_n)(1+\xi_2^2+\cdots+\xi_n^2)^{\frac{1}{2}n-1},$$

$$\frac{1-yy'}{(1-2yv'+yy')^{\frac{n}{2}}}=\frac{x_1(1+xx'+2x_1)^{\frac{n}{2}-1}(1+\xi_2^2+\cdots+\xi_n^2)^{\frac{n}{2}}}{2^{n-2}(x_1^2+(\xi_2+x_2)^2+\cdots+(\xi_n+x_n)^2)^{\frac{n}{2}}},$$

$$\dot{v}=\left(\frac{2}{1+\xi_2^2+\cdots+\xi_n^2}\right)^{n-1}d\xi_2\cdots d\xi_n.$$

由 (4) 及 (5) 得

$$\begin{aligned}\Psi(x)=&(1+xx'+2x_1)^{1-\frac{n}{2}}\Phi(y)\\=&(1+xx'+2x_1)^{1-\frac{n}{2}}\frac{1}{\omega_{n-1}}\int\cdots\int_{vv'=1}\frac{1-yy'}{(1-2yv'+yy')^{\frac{n}{2}}}\Phi(v)\dot{v}\\=&(1+xx'+2x_1)^{1-\frac{n}{2}}\frac{1}{\omega_{n-1}}\\&\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{x_1(1+xx'+2x_1)^{\frac{n}{2}-1}(1+\xi_2^2+\cdots+\xi_n^2)^{\frac{n}{2}}\Psi(0,\xi_2,\cdots,\xi_n)}{2^{n-2}(x_1^2+(\xi_2+x_2)^2+\cdots+(\xi_n+x_n)^2)^{n/2}}\\&\times(1+\xi_2^2+\cdots+\xi_n^2)^{\frac{1}{2}n-1}\left(\frac{2}{1+\xi_2^2+\cdots+\xi_n^2}\right)^{n-1}d\xi_2\cdots d\xi_n\\=&\frac{1}{\omega_{n-1}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{2x_1\Psi(0,\xi_2,\cdots,\xi_n)}{(x_1^2+(\xi_2+x_2)^2+\cdots+(\xi_n+x_n)^2)^{\frac{n}{2}}}d\xi_2\cdots d\xi_n.\end{aligned}$$

这是 "上半" 空间的 Laplace 方程的 Poisson 公式.

3.4 两球相切, 球串

两球

$$(\xi,\eta_1,\eta_2),\quad(\xi^*,\eta_1^*,\eta_2^*)$$

相切的条件是什么? 当然是

$$\left[\left(\frac{\xi}{2\eta_1}-\frac{\xi^*}{2\eta_1^*}\right)\left(\frac{\xi}{2\eta_1}-\frac{\xi^*}{2\eta_1^*}\right)'\right]^{\frac{1}{2}}=\sqrt{\frac{\xi\xi'-4\eta_1\eta_2}{4\eta_1^2}}\pm\sqrt{\frac{\xi^*\xi^{*'}-4\eta_1^*\eta_2^*}{4\eta_1^{*2}}},$$

即

$$(2\eta_1\eta_2^*+2\eta_2\eta_1^*-\xi\xi^{*'})^2=(\xi\xi'-4\eta_1\eta_2)(\xi^*\xi^{*'}-4\eta_1^*\eta_2^*),$$

也就是行列式

$$\left|\begin{pmatrix}\xi, & \eta_1, & \eta_2\\ \xi^*, & \eta_1^*, & \eta_2^*\end{pmatrix}J^{-1}\begin{pmatrix}\xi, & \eta_1, & \eta_2\\ \xi^*, & \eta_1^*, & \eta_2^*\end{pmatrix}'\right|=0. \tag{1}$$

这建议, 我们应当研究二行二列的方阵

$$\begin{pmatrix}\xi, & \eta_1, & \eta_2\\ \xi^*, & \eta_1^*, & \eta_2^*\end{pmatrix}J^{-1}\begin{pmatrix}\xi, & \eta_1, & \eta_2\\ \xi^*, & \eta_1^*, & \eta_2^*\end{pmatrix}'=S. \tag{2}$$

我们定义球串：由形如

$$\lambda(\xi,\eta_1,\eta_2)+\mu(\xi^*,\eta_1^*,\eta_2^*)$$

的球所组成的集合称为球串.

球串可以用 $2\times(n+2)$ 的矩阵

$$X=\begin{pmatrix}\xi, & \eta_1, & \eta_2\\ \xi^*, & \eta_1^*, & \eta_2^*\end{pmatrix}$$

来表它. 如果有一二行二列的方阵 Q 使

$$QX=Y,$$

则 X,Y 表示同一球串.

由于 J^{-1} 的标签是 $n+1$ 正, 1 负, 所以 S 有三种可能性: (i) 定正, (ii) 降秩, (iii) 标签一正一负, 所对应的球串各定义为椭圆的、抛物的及双曲的.

3.5　两球正交, 球族

两球

$$(\xi,\eta_1,\eta_2),\quad (\xi^*,\eta_1^*,\eta_2^*)$$

的夹角为 θ, 则

$$\left(\frac{\xi}{2\eta_1}-\frac{\xi^*}{2\eta_1^*}\right)\left(\frac{\xi}{2\eta_1}-\frac{\xi^*}{2\eta_1^*}\right)=\frac{\xi\xi'-4\eta_1\eta_2}{4\eta_1^2}+\frac{\xi^*\xi^{*'}-4\eta_1^*\eta_2^*}{4\eta_1^{*2}}$$
$$-2\sqrt{\frac{\xi\xi'-4\eta_1\eta_2}{4\eta_1^2}}\sqrt{\frac{\xi^*\xi^{*'}-4\eta_1^*\eta_2^*}{4\eta_1^{*2}}}\cos\theta,$$

也就是

$$\cos\theta=\frac{\xi\xi^{*'}-2\eta_1\eta_2^*-2\eta_2\eta_1^*}{\sqrt{(\xi\xi'-4\eta_1\eta_2)(\xi^*\xi^{*'}-4\eta_1^*\eta_2^*)}}$$
$$=\frac{(\xi,\eta_1,\eta_2)J^{-1}(\xi^*,\eta_1^*,\eta_2^*)'}{\sqrt{(\xi,\eta_1,\eta_2)J^{-1}(\xi,\eta_1,\eta_2)'(\xi^*,\eta_1^*,\eta_2^*)J^{-1}(\xi^*,\eta_1^*,\eta_2^*)'}}.$$

二球正交的条件是

$$(\xi,\eta_1,\eta_2)J^{-1}(\xi^*,\eta_1^*,\eta_2^*)'=0. \tag{1}$$

定义　与一定球正交的诸球所成的集合称为球族.

因之, 依被正交球是实球、点球与虚球, 而得出双曲、抛物、椭圆的三族.

3.6 保 角 映 象

本节开始讲些更一般的结果.

如果有一变形

$$y = y(x),$$

使

$$dydy' = \frac{1}{\lambda^2}dxdx', \quad \lambda = \lambda(x), \tag{1}$$

并且把域 D_x 变为域 D_y, 这个变形称为把 D_x 变为 D_y 的保角映象. 关于 Laplace 算子, 我们有以下的性质:

定理 1 如果变换 $y = y(x)$ 适合于 (1), 而

$$Y(y) = X(x)\lambda^{\frac{1}{2}n-1}, \tag{2}$$

则

$$\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2} = \lambda^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}. \tag{3}$$

证 (1) 由 (1) 推得

$$\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\frac{\partial y_i}{\partial x_q} = \frac{1}{\lambda^2}\delta_{pq}, \quad \sum_{i=1}^{n}\frac{\partial x_i}{\partial y_p}\frac{\partial x_i}{\partial y_q} = \lambda^2\delta_{pq}, \tag{4}$$

而且有

$$\frac{\partial x_i}{\partial y_p} = \lambda^2\frac{\partial y_p}{\partial x_i}. \tag{5}$$

(2) 微分 (4) 式得

$$\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_s}\cdot\frac{\partial y_i}{\partial x_q} + \sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_s} = \delta_{pq}\frac{\partial\lambda^{-2}}{\partial x_s}, \tag{6}$$

交换足码 s 与 q 得

$$\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_q}\cdot\frac{\partial y_i}{\partial x_s} + \sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_s} = \delta_{ps}\frac{\partial\lambda^{-2}}{\partial x_q}. \tag{7}$$

(6) 与 (7) 相加得

$$\frac{\partial}{\partial x_p}\left(\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_s}\frac{\partial y_i}{\partial x_q}\right) + 2\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\frac{\partial^2 y_i}{\partial x_q\partial x_s} = \delta_{pq}\frac{\partial\lambda^{-2}}{\partial x_s} + \delta_{ps}\frac{\partial\lambda^{-2}}{\partial x_q},$$

即得

$$2\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_s} = \delta_{pq}\frac{\partial\lambda^{-2}}{\partial x_s} + \delta_{ps}\frac{\partial\lambda^{-2}}{\partial x_q} - \delta_{sq}\frac{\partial\lambda^{-2}}{\partial x_p}. \tag{8}$$

乘以 $\dfrac{\partial x_p}{\partial y_j}$, 并且对 p 求和得

$$2\frac{\partial^2 y_j}{\partial x_q\partial x_s}=\frac{\partial\lambda^{-2}}{\partial x_s}\frac{\partial x_q}{\partial y_j}+\frac{\partial\lambda^{-2}}{\partial x_q}\frac{\partial x_s}{\partial y_j}-\delta_{sq}\frac{\partial\lambda^{-2}}{\partial y_j}. \tag{9}$$

(3) 命 $q=s$, 并对 q 求和得

$$2\sum_{q=1}^{n}\frac{\partial^2 y_j}{\partial x_q^2}=(2-n)\frac{\partial\lambda^{-2}}{\partial y_j}. \tag{10}$$

(4) 再求 (8) 式对 x_t 的偏微商得

$$\begin{aligned}&2\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_t}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_s}+2\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\cdot\frac{\partial^3 y_i}{\partial x_q\partial x_s\partial x_t}\\=&\delta_{pq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_t}+\delta_{ps}\frac{\partial^2\lambda^{-2}}{\partial x_q\partial x_t}-\delta_{sq}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_t}.\end{aligned} \tag{11}$$

交换足码 p 与 t 得

$$\begin{aligned}&2\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_t}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_s}+2\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_t}\frac{\partial^3 y_i}{\partial x_q\partial x_s\partial x_p}\\=&\delta_{tq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_p}+\delta_{ts}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_q}-\delta_{sq}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_t}.\end{aligned} \tag{12}$$

两式相加得

$$\begin{aligned}&4\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_t}\frac{\partial^2 y_i}{\partial x_q\partial x_s}+2\left(\sum_{t=1}^{n}\frac{\partial y_i}{\partial x_p}\frac{\partial^3 y_i}{\partial x_q\partial x_s\partial x_t}+\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_t}\frac{\partial^3 y_i}{\partial x_q\partial x_s\partial x_p}\right)\\=&\delta_{pq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_t}+\delta_{ps}\frac{\partial^2\lambda^{-2}}{\partial x_q\partial x_t}+\delta_{tq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_p}+\delta_{ts}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_q}-2\delta_{sq}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_t}.\end{aligned} \tag{13}$$

利用

$$\frac{\partial^2}{\partial x_q\partial x_s}\left(\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\frac{\partial y_i}{\partial x_t}\right)=\frac{\partial^2\lambda^{-2}}{\partial x_q\partial x_s}\delta_{pt} \tag{14}$$

及

$$\begin{aligned}\frac{\partial^2}{\partial x_q\partial x_s}\left(\sum_{i=1}^{n}\frac{\partial y_i}{\partial x_p}\frac{\partial y_i}{\partial x_t}\right)=&\sum_{i=1}^{n}\left(\frac{\partial y_i}{\partial x_p}\frac{\partial^3 y_i}{\partial x_q\partial x_s\partial x_t}+\frac{\partial y_i}{\partial x_t}\cdot\frac{\partial^3 y_i}{\partial x_q\partial x_s\partial x_p}\right.\\&\left.+\frac{\partial^2 y_i}{\partial x_p\partial x_q}\cdot\frac{\partial^2 y_i}{\partial x_s\partial x_t}+\frac{\partial^2 y_i}{\partial x_p\partial x_s}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_t}\right),\end{aligned} \tag{15}$$

由 (13), (14) 与 (15) 得

$$4\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_t}\frac{\partial^2 y_i}{\partial x_q\partial x_s}+2\left(\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_q}\delta_{pt}-\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_q}\right.$$

$$\left.\cdot\frac{\partial^2 y_i}{\partial x_s\partial x_t}-\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_s}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_t}\right)$$

$$=\delta_{pq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_t}+\delta_{ps}\frac{\partial^2\lambda^{-2}}{\partial x_q\partial x_t}+\delta_{tq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_p}+\delta_{ts}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_q}-2\delta_{sq}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_t},$$

即

$$4\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_t}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_s}-2\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_q}\cdot\frac{\partial^2 y_i}{\partial x_s\partial x_t}-2\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p\partial x_s}\cdot\frac{\partial^2 y_i}{\partial x_q\partial x_t}$$

$$=\delta_{pq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_t}+\delta_{ps}\frac{\partial^2\lambda^{-2}}{\partial x_q\partial x_t}+\delta_{tq}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_p}+\delta_{ts}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_q}$$

$$-2\delta_{sq}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_t}-2\delta_{pt}\frac{\partial^2\lambda^{-2}}{\partial x_s\partial x_q}. \tag{16}$$

(5) 在 (16) 中取 $t=p, s=q$ 得

$$4\sum_{i=1}^{n}\frac{\partial^2 y_i}{\partial x_p^2}\cdot\frac{\partial^2 y_i}{\partial x_q^2}-4\sum_{i=1}^{n}\left(\frac{\partial^2 y_i}{\partial x_p\partial x_q}\right)^2=4\delta_{pq}\frac{\partial^2\lambda^{-2}}{\partial x_p\partial x_q}-2\frac{\partial^2\lambda^{-2}}{\partial x_p^2}-2\frac{\partial^2\lambda^{-2}}{\partial x_q^2}.$$

对 p,q 求和得

$$\sum_{i=1}^{n}\left[\left(\sum_{p}\frac{\partial^2 y_i}{\partial x_p^2}\right)^2-\sum_{p,q}\left(\frac{\partial^2 y_i}{\partial x_p\partial x_q}\right)^2\right]=(1-n)\sum_{p=1}^{n}\frac{\partial^2\lambda^{-2}}{\partial x_p^2}. \tag{17}$$

(6) 由 (9) 可知

$$4\left(\frac{\partial^2 y_i}{\partial x_p\partial x_q}\right)^2=\left(\frac{\partial\lambda^{-2}}{\partial x_p}\frac{\partial x_q}{\partial y_i}+\frac{\partial\lambda^{-2}}{\partial x_q}\frac{\partial x_p}{\partial y_i}-\delta_{pq}\frac{\partial\lambda^{-2}}{\partial y_i}\right)^2,$$

由此

$$4\sum_{i}\sum_{p,q}\left(\frac{\partial^2 y_i}{\partial x_p\partial x_q}\right)^2=4\sum_{p,q}\sum_{i}\left(\frac{\partial^2 y_i}{\partial x_p\partial x_q}\right)^2$$

$$=\sum_{p,q}\sum_{i}\left[\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2\left(\frac{\partial x_q}{\partial y_i}\right)^2+\left(\frac{\partial\lambda^{-2}}{\partial x_q}\right)^2\left(\frac{\partial x_p}{\partial y_i}\right)^2\right.$$

$$
\begin{aligned}
&+\delta_{pq}\left(\frac{\partial\lambda^{-2}}{\partial y_i}\right)^2+2\frac{\partial\lambda^{-2}}{\partial x_p}\frac{\partial\lambda^{-2}}{\partial x_q}\frac{\partial x_q}{\partial y_i}\frac{\partial x_p}{\partial y_i}\\
&-2\frac{\partial\lambda^{-2}}{\partial x_p}\cdot\frac{\partial\lambda^{-2}}{\partial y_i}\delta_{pq}\frac{\partial x_q}{\partial y_i}-2\frac{\partial\lambda^{-2}}{\partial x_q}\frac{\partial\lambda^{-2}}{\partial y_i}\delta_{pq}\frac{\partial x_p}{\partial y_i}\Bigg]\\
=&2n\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2+n\sum_i\left(\frac{\partial\lambda^{-2}}{\partial y_i}\right)^2+2\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2\\
&-4\sum_i\left(\frac{\partial\lambda^{-2}}{\partial y_i}\right)^2=2(n+1)\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2\\
&+(n-4)\sum_i\left(\frac{\partial\lambda^{-2}}{\partial y_i}\right)^2=(3n-2)\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2,
\end{aligned}\tag{18}
$$

这里用了

$$
\begin{aligned}
\sum_{i=1}^n\left(\frac{\partial\lambda^{-2}}{\partial y_i}\right)^2&=\sum_{i=1}^n\left(\sum_j\frac{\partial\lambda^{-2}}{\partial x_j}\frac{\partial x_j}{\partial y_i}\right)^2\\
&=\sum_{j,k}\frac{\partial\lambda^{-2}}{\partial x_j}\frac{\partial\lambda^{-2}}{\partial x_k}\sum_i\frac{\partial x_j}{\partial y_i}\frac{\partial x_k}{\partial y_i}=\lambda^2\sum_{j=1}^n\left(\frac{\partial\lambda^{-2}}{\partial x_j}\right)^2.
\end{aligned}\tag{19}
$$

(7) 将 (10) 与 (18) 代进 (17) 得 (并且 (19))

$$
\sum_{j=1}^n\left[\left(1-\frac{n}{2}\right)\frac{\partial\lambda^{-2}}{\partial y_j}\right]^2-\frac{3n-2}{4}\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2=(1-n)\sum_p\frac{\partial^2\lambda^{-2}}{\partial x_p^2},
$$

即

$$
\left[\left(1-\frac{n}{2}\right)^2-\frac{3n-2}{4}\right]\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2=(1-n)\sum_p\frac{\partial^2\lambda^{-2}}{\partial x_p^2},
$$

$$
\frac{(1-n)(6-n)}{4}\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2=(1-n)\sum_p\frac{\partial^2\lambda^{-2}}{\partial x_p^2},
$$

即

$$
(6-n)\lambda^2\sum_p\left(\frac{\partial\lambda^{-2}}{\partial x_p}\right)^2=4\sum_p\frac{\partial^2\lambda^{-2}}{\partial x_p^2},
$$

也就是

$$
\frac{n}{2}\sum_p\left(\frac{\partial\lambda}{\partial x_p}\right)^2=\lambda\sum_p\frac{\partial^2\lambda}{\partial x_p^2}.\tag{20}
$$

(8) 由

$$\frac{\partial Y}{\partial y_i}=\sum_{j=1}^{n}\frac{\partial Y}{\partial x_j}\frac{\partial x_j}{\partial y_i}$$

及

$$\frac{\partial^2 Y}{\partial y_i^2}=\sum_{j,k=1}^{n}\frac{\partial^2 Y}{\partial x_j\partial x_k}\cdot\frac{\partial x_j}{\partial y_i}\cdot\frac{\partial x_k}{\partial y_i}+\sum_{j=1}^{n}\frac{\partial Y}{\partial x_j}\frac{\partial^2 x_j}{\partial y_i^2}.$$

由 (10) 可知

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2}&=\lambda^2\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial x_i^2}+\sum_{j=1}^{n}\frac{\partial Y}{\partial x_j}\sum_{i=1}^{n}\frac{\partial^2 x_j}{\partial y_i^2}\\&=\lambda^2\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial x_i^2}+(2-n)\lambda\sum_{j=1}^{n}\frac{\partial Y}{\partial x_j}\frac{\partial\lambda}{\partial x_j}\\&=\lambda^n\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\lambda^{2-n}\frac{\partial Y}{\partial x_i}\right).\end{aligned}$$

(9) 利用恒等式

$$\frac{\partial}{\partial x_i}\left(\Omega^2\frac{\partial}{\partial x_i}(\Omega^{-1}\Phi)\right)=\Omega\frac{\partial^2\Phi}{\partial x_i^2}-\Phi\frac{\partial^2\Omega}{\partial x_i^2},$$

推得

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2}&=\lambda^n\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\lambda^{2-n}\frac{\partial(\lambda^{\frac{n}{2}-1}X)}{\partial x_i}\right)\\&=\sum_{i=1}^{n}\lambda^n\left(\lambda^{1-\frac{n}{2}}\frac{\partial^2 X}{\partial x_i^2}-X\frac{\partial^2\lambda^{1-\frac{n}{2}}}{\partial x_i^2}\right)\\&=\lambda^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}-X\lambda^n\sum_{i=1}^{n}\frac{\partial^2\lambda^{1-\frac{n}{2}}}{\partial x_i^2}\\&=\lambda^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}-X\left(1-\frac{n}{2}\right)\lambda^{\frac{n}{2}-1}\left(\lambda\sum_i\frac{\partial^2\lambda}{\partial x_i^2}-\frac{n}{2}\sum_i\left(\frac{\partial\lambda}{\partial x_i}\right)^2\right),\end{aligned}$$

由 (20) 可知最后一项等于 0. 因此得出

$$\sum_{i=1}^{n}\frac{\partial^2 Y}{\partial y_i^2}=\lambda^{\frac{n}{2}+1}\sum_{i=1}^{n}\frac{\partial^2 X}{\partial x_i^2}.$$

这就是本定理的结论.

附记 在定理的证明中也带便地证明了

$$\lambda^{1-\frac{n}{2}}, \quad \lambda^{1-\frac{n}{2}} y_i, \quad i=1,\cdots,n$$

都是调和函数.

问题 1 任何一个把单位球变为其内部的保角变换是否把两点间的非欧距离愈变愈短?

问题 2 如果一个把单位球变为其内部而且把原点变为原点的保角变换是否把同心球也变为其内部?

第 4 讲　Lorentz 群

4.1　换基本方阵

以往我们定义

$$J=\begin{pmatrix} I^{(n)} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \tag{1}$$

而研究适合于

$$MJM'=J \tag{2}$$

的 M 所成的群.

由于

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

如命

$$P=\begin{pmatrix} I^{(n)} & 0 \\ 0 & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{pmatrix} M \begin{pmatrix} I^{(n)} & 0 \\ 0 & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{pmatrix}^{-1}, \tag{3}$$

则得

$$P\begin{pmatrix} I^{(n+1)} & 0 \\ 0 & -1 \end{pmatrix}P'=\begin{pmatrix} I^{(n+1)} & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

关于式 (3) 可以写得更具体:

$$P=\begin{pmatrix} M_1 & \frac{1}{2}(u_1'-u_2') & \frac{1}{2}(u_1'+u_2') \\ v_1-v_2 & \frac{1}{2}(a-b-c+d) & \frac{1}{2}(a+b-c-d) \\ v_1+v_2 & \frac{1}{2}(a-b+c-d) & \frac{1}{2}(a+b+c+d) \end{pmatrix},$$

其中

$$M=\begin{pmatrix} M_1 & u_1' & u_2' \\ v_1 & a & b \\ v_2 & c & d \end{pmatrix}.$$

在今后相当长的一段时期内, 我们研究适合于 (4) 的 P 所成的群, 这群以 $L(n+1,1)$ 表之, 称为 $(n+1,1)$ 型 Lorentz 群.

由 (4) 式可知

$$(\det P)^2=1,$$

即 P 之行列式之值等于 ±1, 行列式之值等于 1 的 P 成一子群, 以 $L^+_{(n+1,1)}$ 表之, 行列式之值等于 -1 的 P 构成一子集合, 以 $L^-_{(n+1,1)}$ 表之. 特别如 $[1,\cdots,1,-1]$ 就是一个行列式等于 -1 的 $L_{(n+1,1)}$ 方阵, 且 $L^-_{(n+1,1)}$ 由 $L^+_{(n+1,1)}$ 中方阵乘以 $[1,\cdots,1,-1]$ 构成, 即

$$L^-_{(n+1,1)}=[1,\cdots,1,-1]L^+_{(n+1,1)}.$$

所以群 $L_{(n+1,1)}$ 是由 $L^+_{(n+1,1)}$ 添加任一行列式等于 -1 的方阵而生成的, 而且

$$L_{(n+1,1)}=L^+_{(n+1,1)}\bigcup L^-_{(n+1,1)}=L^+_{(n+1,1)}\bigcup[1,\cdots,1,-1]L^+_{(n+1,1)}. \tag{5}$$

又如果把 P 的元素写成为

$$(a_{ij})_{1\leqslant i,j\leqslant n+2},$$

则适合于

$$a_{n+2,n+2}>0 \tag{6}$$

的方阵也成一群, 以 $L_{+(n+1,1)}$ 表之. 今往证明适合 (6) 的方阵成一群, 命

$$B=(b_{ij})_{1\leqslant i,j\leqslant n+2},\quad b_{n+2,n+2}>0,$$

及

$$C=AB,$$

则

$$C_{n+2,n+2}=\sum_{i=1}^{n+1}a_{n+2,i}b_{i,n+2}+a_{n+2,n+2}b_{n+2,n+2}.$$

由关系 (4) 已知

$$\sum_{i=1}^{n+1}a^2_{n+2,i}-a^2_{n+2,n+2}=-1,\quad \sum_{i=1}^{n+1}a^2_{n+2,i}<a^2_{n+2,n+2},$$
$$\sum_{i=1}^{n+1}b^2_{i,n+2}-b^2_{n+2,n+2}=-1,\quad \sum_{i=1}^{n+1}b^2_{i,n+2}<b^2_{n+2,n+2},$$

由 Schwarz 不等式

$$\left|\sum_{i=1}^{n+1}a_{n+2,i}b_{i,n+2}\right|\leqslant\sqrt{\sum_{i=1}^{n+1}a^2_{n+2,i}\sum_{i=1}^{n+1}b^2_{i,n+2}}<a_{n+2,n+2}b_{n+2,+2},$$

即得所证.

适合于

$$a_{n+2,n+2} < 0 \tag{7}$$

的元素构成一子集合, 以 $L_{-(n+1,1)}$ 表之, 也显然有 $[1,\cdots,1,-1] \in L_{-(n+1,1)}$, 且

$$L_{-(n+1,1)} = [1,\cdots,1,-1]L_{+(n+1,1)},$$

而且

$$L_{(n+1,1)} = L_{+(n+1,1)} \bigcup L_{-(n+1,1)} = L_{+(n+1,1)} \bigcup [1,\cdots,1,-1]L_{+(n+1,1)}.$$

即属于 $L^+_{(n+1,1)}$ 又属于 $L_{+(n+1,1)}$ 的元素成一群, 以 $L^+_{+(n+1,1)}$ 表之, 即

$$L^+_{+(n+1,1)} = L^+_{(n+1,1)} \bigcap L_{+(n+1,1)}.$$

不难证明

$$\begin{aligned} L^+_{(n+1,1)} &= L^+_{+(n+1,1)} \bigcup L^+_{-(n+1,1)} \\ &= L^+_{+(n+1,1)} \bigcup [-1,1,\cdots,1,-1]L^+_{+(n+1,1)}, \end{aligned}$$

而

$$L_{(n+1,1)} = L^+_{+(n+1,1)} \bigcup L^+_{-(n+1,1)} \bigcup L^-_{+(n+1,1)} \bigcup L^-_{-(n+1,1)},$$

其中

$$\begin{aligned} L^+_{-(n+1,1)} &= [-1,1,\cdots,1,-1]L^+_{+(n+1,1)}, \\ L^-_{+(n+1,1)} &= [1,\cdots,1,-1,1]L^+_{+(n+1,1)}, \\ L^-_{-(n+1,1)} &= [1,\cdots,1,+1,-1]L^+_{+(n+1,1)}. \end{aligned}$$

注意: 条件 (6) 反映在原符号上是

$$a+b+c+d > 0.$$

4.2 演出元素

置换变换: 固定 $i,j(i \neq j, 1 \leqslant i,j \leqslant n+1)$, 由

$$y_i = x_j, \quad y_j = x_i, \quad y_k = x_k, \quad k \neq i,j; 1 \leqslant k \leqslant n+2 \tag{1}$$

的变形称为置换变形, 以

$$P_{ij}(1 \leqslant i,j \leqslant n+1)$$

表之. $P_{ij}P$ 表之 P 的第 i 行. 第 j 行互换, PP_{ij} 表示 P 的第 i 列、第 j 列互换, 这变换属于 $L^{-}_{+(n+1,1)}$.

如果固定 $i,j(i\neq j,1\leqslant i,j\leqslant n+1)$, 由

$$y_i=x_j,\quad y_j=-x_i,\quad y_k=x_k,\quad k\neq i,j;1\leqslant k\leqslant n+2$$

的变形以

$$Q_{ij}$$

表之, 则 Q_{ij} 属于 $L^{+}_{+(n+1,1)}$.

变形

$$R_{12}(\theta)=\begin{pmatrix}\cos\theta & \sin\theta & 0 & \cdots & 0\\ -\sin\theta & \cos\theta & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}$$

称为旋转, 它属于 $L^{+}_{+(n+1,1)}$.

$$R_{ij}(\theta)=Q_{i1}Q_{j2}R_{12}Q_{j2}^{-1}Q_{i1}^{-1},\quad i,j=1,2,\cdots,n+1$$

所表示之变形是

$$y_i=\cos\theta x_i+\sin\theta x_j,\quad y_j=-\sin\theta x_i+\cos\theta x_j,$$

其他的 $x_k(k\neq i,j)$ 不变, 这类的变形统称为旋转. 又

$$H_1(\psi)=\begin{pmatrix}\cos h\psi & 0 & \cdots & 0 & \sin h\psi\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ \sin h\psi & 0 & \cdots & 0 & \cos h\psi\end{pmatrix}$$

称为双曲旋转.

$$H_i(\psi)=Q_{i1}H_1Q_{i1}^{-1}$$

也称为双曲旋转.

定理 1　$L^{+}_{+(n+1,1)}$ 由置换、旋转、双曲旋转所演成的.

证　(1) 先由极简单的性质出发: 令

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\cos h\psi & \sin h\psi\\ \sin h\psi & \cos h\psi\end{pmatrix}=\begin{pmatrix}* & *\\ c' & *\end{pmatrix},$$

由计算可知

$$c' = c\cos h\psi + d\sin h\psi.$$

如果 $d \neq 0$, $|c| \leqslant |d|$, 则由

$$c\cos h\psi + d\sin h\psi = 0,$$

亦即由

$$\tan h\psi = -\frac{c}{d}$$

可以解出 ψ, 即有 ψ 使 $c' = 0$.

(2) 命 A 是 $L^+_{+(n+1,1)}$ 中任一方阵, 记

$$A = \begin{pmatrix} A_1^{(n+1)} & \beta \\ \alpha & a_{n+2,n+2} \end{pmatrix},$$

则由 $A[1,\cdots,1,-1]A' = [1,\cdots,1,-1]$ 可知

$$\alpha\alpha' + 1 = a^2_{n+2,n+2},$$

因此

$$a_{n+2,n+2} \neq 0, \quad |a_{n+2,j}| \leqslant |a_{n+2,n+2}|, \quad j = 1, 2, \cdots, n+1.$$

所以有 $H_1(\psi_1)$ 使

$$AH_1(\psi_1)$$

的 $(n+2,1)$ 元素为 0, $AH_1(\psi_1)$ 仍旧属于 $L^+_{+(n+1,1)}$. 再乘以合适的 $H_2(\psi_2)$ 使 $(n+2,2)$ 元素为 0, 等等, 即可以乘以双曲旋转方阵使 A 变为

$$\begin{pmatrix} \widetilde{A}_1^{(n+1)} & \widetilde{\beta} \\ 0 & \widetilde{a}_{n+2,n+2} \end{pmatrix}$$

的形式, 由于它适合 (1.4) 式, 因此 $\widetilde{\beta} = 0$, 即

$$\widetilde{a}_{1,n+2} = \cdots = \widetilde{a}_{n+1,n+2} = 0.$$

所以 A 变为

$$B = \begin{pmatrix} B_1^{(n+1)} & 0 \\ 0 & a \end{pmatrix}, \quad B_1B_1' = I^{(n+1)}, \quad a^2 = 1,$$

由于 B 属于 $L^+_{+(n+1,1)}$, 所以 $a = 1$, $\det B_1 = 1$. 因而 B_1 是一行列式等于 1 的正交方阵, 可知它是旋转之积 (读者如不知此结果, 仍可由上法逐步得出之).

说得更深刻些, $L^{+}_{+(n+1,1)}$ 可由置换 Q_{ij}, $R_{12}(\theta)$, $H_1(\psi)$ 而演出之, 置换共有 $\frac{1}{2}n(n+1)$ 个, 但注意它们仍然可以用

$$Q_{12} \quad 及 \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ (-1)^n & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

演出之, 但 $Q_{12} = R_{12}\left(\frac{\pi}{2}\right)$, 因此一共有三种元素即可演出 $L^{+}_{+(n+1,1)}$.

问题　演出元素还能再减少否?

定理 2　$L_{(n+1,1)}$ 的元素可以由 Q_{ij}, $R_{12}(\theta)$, $H_1(\psi)$, $[1,\cdots,1,-1]$, $[1,\cdots,1,-1,-1]$ 演出之.

回到原来的形式, 对应于 R_{12} 有

$$\begin{aligned} y_1 &= \cos\theta x_1 + \sin\theta x_2, \\ y_2 &= -\sin\theta x_1 + \cos\theta x_2, \\ y_i &= x_i, \quad i = 3, 4, \cdots, n+2. \end{aligned}$$

其次对应于

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ (-1)^n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

我们有

$$u_1 = 2e_n, \quad u_2 = -2e_n, \quad v_1 = \frac{(-1)^n}{2}e_1,$$
$$v_2 = -\frac{(-1)^n}{2}e_1, \quad a = b = c = d = \frac{1}{2},$$

即

$$\begin{aligned} y_1 &= (-1)^n \frac{xx' - 1}{-4x_n + xx' + 1}, \\ y_i &= \frac{2x_{i-1}}{-4x_n + xx' + 1}, \quad i = 2, 3, \cdots, n. \end{aligned}$$

而对应于 $H_1(\psi)$, 我们有

$$y_1 = \frac{(\cos h\psi)x_1}{(\sin h\psi)x_1 + \frac{1}{2}(1 - \cos h\psi)xx' + \frac{1}{2}(1 + \cos h\psi)},$$

$$y_i = \frac{x_i}{(\sin h\psi)x_1 + \frac{1}{2}(1-\cos h\psi)xx' + \frac{1}{2}(1+\cos h\psi)},$$

$$y_n = \frac{x_n}{(\sin h\psi)x_1 + \frac{1}{2}(1-\cos h\psi)xx' + \frac{1}{2}(1+\cos h\psi)}.$$

4.3 正交相似

为了便于引进和解决 Lorentz 相似的问题, 我们重复一下正交相似的概念和处理方法.

我们现在考虑 m 行列的正交方阵, 也就是考虑适合于

$$TT' = I \tag{1}$$

的实方阵 T.

命 A, B 是两个正交方阵, 如果有一个正交方阵 T 存在使

$$TAT^{-1} = B, \tag{2}$$

则 A, B 称为正交相似.

正交相似的几何意义是: 在一个正交系统有一个正交变换 A, 换为另一个正交系统, 它和原系统关系为 T, 则在新系统下, 这正交变换的方阵是 TAT^{-1}.

最熟悉的例子是 $m=3$. 任何一个行列式等于 1 的正交变换一定可以选择系统使它变为绕 z 轴的旋转, 也就是有 T 使

$$TAT^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

我们现在将证明, 任何一个正交方阵一定正交相似于

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dot{+} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix}$$
$$\dot{+} 1 \dot{+} \cdots \dot{+} 1 \dot{+} (-1) \dot{+} \cdots \dot{+} (-1), \tag{3}$$

这里 $0 < \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_v < 2\pi$.

这是熟知的结果, 但我们依旧写下它的证明作为今后的楷模.

(1) 先证: 有了几个互相正交的单位矢量

$$v_1, \cdots, v_r \quad (r < m),$$

即

$$v_i v_j' = \delta_{ij},$$

我们一定还可以添上一个 v_{r+1}, 即

$$v_{r+1}v_i' = 0, \quad v_{r+1}v_{r+1}' = 1.$$

这证明是十分简单的. 取一与 $v_1, \cdots, v_r$ 线性无关的矢量 u. 命

$$u_{r+1} = u - c_1 v_1 - \cdots - c_r v_r, c_\nu = uv_\nu',$$

则显然有

$$u_{r+1}v_\nu' = uv_\nu' - c_\nu = 0.$$

由于 u_{r+1} 非 0 矢量, 所以

$$u_{r+1}u_{r+1}' = \xi > 0.$$

而 $v_{r+1} = \dfrac{1}{\sqrt{\xi}}u_{r+1}$ 即合所求.

(2) 正交方阵的特征根的绝对值等于 1.

如果 λ 是 A 的特征根则其对应的特征矢量是 z, 即

$$zA = \lambda z.$$

由于 A 是实方阵, 所以

$$\bar{z}A = \bar{\lambda}\,\bar{z}.$$

因此

$$z\bar{z}' = zAA'\bar{z}' = |\lambda|^2 z\bar{z}',$$

而 $z\bar{z}' \neq 0$, 因此 $|\lambda|^2 = 1$.

(3) 如果 A 有一复特征根 $e^{i\theta}(\neq \pm 1)$, 而

$$zA = e^{i\theta}z,$$

把 z 分为虚实部分 $z = x + yi$, 则

$$xA = \cos\theta x - \sin\theta y, \quad yA = \sin\theta x + cos\theta y,$$

即

$$\begin{pmatrix} x \\ y \end{pmatrix} A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

二行二列的方阵

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} xx' & xy' \\ yx' & yy' \end{pmatrix} = \begin{pmatrix} s & t \\ t & u \end{pmatrix}$$

是定正的, 而且

$$\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix} AA' \begin{pmatrix} x \\ y \end{pmatrix}'$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}'\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}',$$

即

$$\begin{pmatrix} s & t \\ t & u \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} s & t \\ t & u \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

由此推得 $t = 0, s = u$. 命

$$v_1 = \frac{1}{\sqrt{s}}x, \quad v_2 = \frac{1}{\sqrt{s}}y,$$

则得

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

而 v_1, v_2 是互相正交的二单位矢量.

由 (1) 可以作一方阵 T 以 v_1, v_2 为其第一、二行, 则

$$TAT^{-1} = TAT' = \begin{pmatrix} \cos\theta & -\sin\theta & a_{13} & \cdots & a_{1m} \\ \sin\theta & \cos\theta & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}$$

$\left(这里用了 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right)$, 再由 TAT' 的正交性可得

$$a_{13} = \cdots = a_{1m} = a_{23} = \cdots = a_{2m} = 0,$$
$$a_{31} = \cdots = a_{m1} = a_{32} = \cdots = a_{m2} = 0.$$

用归纳法即得所求证的结果.

(4) 如果 A 有 1 为特征根, 命 x 是其对应的特征矢量, 即

$$xA = x,$$

不妨假定 $xx'=1$, 作一以 x 为第一行的正交方阵 T, 则

$$TAT^{-1}=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & a_{22} & \cdots & a_{2m}\\ \vdots & \vdots & & \vdots\\ 0 & a_{m2} & \cdots & a_{mm}\end{pmatrix}.$$

同法处理 A 有 -1 为特征根的情况.

附记 二正交方阵正交相似的必要且充分的条件是它们有相同的特征多项式.

4.4 关于非定正二次型

先证明与二次型

$$x_1^2+\cdots+x_{m-1}^2-x_m^2=xFx',\quad F=[1,\cdots,1,-1]$$

有关的几个容易的结果.

(1) 不能有两个线性无关的矢量 y, z 对任何 λ, μ 常使

$$(\lambda y+\mu z)F(\lambda y+\mu z)'=0,$$

即

$$\lambda^2yFy'+2\lambda\mu yFz'+\mu^2zFz'=0,$$

即

$$yFy'=0,\quad yFz'=0,\quad zFz'=0,$$

即

$$y_1^2+\cdots+y_{m-1}^2=y_m^2,$$

$$z_1^2+\cdots+z_{m-1}^2=z_m^2,$$

$$y_1z_1+\cdots+y_{m-1}z_{m-1}=y_mz_m.$$

由此推得

$$(y_1^2+\cdots+y_{m-1}^2)(z_1^2+\cdots+z_{m-1}^2)-(y_1z_1+\cdots+y_{m-1}z_{m-1})^2=0,$$

即

$$\sum_{i<j}(y_iz_j-y_jz_i)^2=0.$$

这与 y, z 的独立性矛盾.

(2) 命 P 是一 l 行 m 列的矩阵 ($l<m$), 则

$$PFP'$$

的标签中最多只有一个负号.

PFP' 是一 l 行 l 列的方阵, 如果标签中有两个负号, 即有 $Q(=Q^{(l)})$ 使

$$QPFP'Q'=\begin{pmatrix}-1 & & \\ & -1 & \\ & & \ddots\end{pmatrix}.$$

命 QP 的第一, 第二行是 y,z, 则

$$yFy'=-1,\quad zFz'=-1,\quad yFz'=0.$$

即

$$\sum_{i=1}^{m-1}y_i^2=y_m^2-1,\quad \sum_{i=1}^{m-1}z_i^2=z_m^2-1,\quad \sum_{i=1}^{m-1}y_iz_i=z_my_m,$$

即

$$\left(\sum_{i=1}^{m-1}y_i^2+1\right)\left(\sum_{i=1}^{m-1}z_i^2+1\right)=\left(\sum_{i=1}^{m-1}y_iz_i\right)^2,$$

同法这是不可能的.

(3) 以上的证明方法实质上给出了以下的结果.

如果 P 的秩等于 l, 则

$$PFP'$$

的标签只有以下三种可能性:

(i) 定正, 即 l 个 $+1$;

(ii) 半定正, 即 $l-1$ 个 $+1$, 一个 0;

(iii) 非定正, 即 $l-1$ 个 $+1$, 一个 -1;

而无其他的可能性.

4.5 Lorentz 相似

在以下几节中命

$$F=\begin{pmatrix}I^{(n+1)} & 0\\ 0 & -1\end{pmatrix}, \tag{1}$$

适合于

$$TFT' = F$$

的方阵简称为 Lorentz 方阵.

命 A, B 是两个 Lorentz 方阵, 如果有一 Lorentz 方阵 T 使

$$TAT^{-1} = B, \tag{2}$$

则 A, B 称为 Lorentz 相似.

我们将证明: 任何一个 Lorentz 方阵一定 Lorentz 等价于以下六种类型的方阵的直和:

$$\begin{pmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{pmatrix}, \quad +1, \quad -1, \quad \begin{pmatrix} \cos h\psi, & \sin h\psi \\ \sin h\psi, & \cos h\psi \end{pmatrix},$$

$$\begin{pmatrix} 0 & 3 & 2\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 2\sqrt{2} & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & -3 & 2\sqrt{2} \\ 1 & 0 & 0 \\ 0 & -2\sqrt{2} & -3 \end{pmatrix}.$$

但最后三种总共只能出现一次.

(1) 在证明这结果之前先证明:

如果 $v_1, \cdots, v_r$ 适合于

$$v_i F v_j' = \delta_{ij}, \quad i, j = 1, \cdots, r,$$

则可以添一 v_{r+1} 使

$$v_{r+1} F v_j' = 0, \quad j = 1, \cdots, r,$$

及

$$v_{r+1} F v_{r+1}' = -1.$$

命

$$u_{r+1} = e_n - \lambda_1 v_1 - \cdots - \lambda_r v_r, \quad \lambda_v = e_n F v_v',$$

则

$$u_{r+1} F v_v' = e_n F v_v' - \lambda_v = 0.$$

再看

$$\begin{aligned} u_{r+1} F u_{r+1}' =& e_n F e_n' - 2\sum_{v=1}^{r} \lambda_v e_n F v_v' + \sum_{i,j} \lambda_i \lambda_j v_i F v_j' \\ =& -1 - 2\sum_{v=1}^{r} \lambda_v^2 + \sum_{v=1}^{r} \lambda_v^2 = -1 - \sum_{v=1}^{r} \lambda_v^2 < 0, \end{aligned}$$

而

$$v_{r+1} = u_{r+1} / \sqrt{1 + \lambda_1^2 + \cdots + \lambda_r^2}$$

即合所求.

如果 $r+1<n$, 则还可以添上 v_{r+2} 使

$$v_{r+2}Fv_\nu'=0,\quad \nu=1,2,\cdots,r+1$$

及

$$v_{r+2}Fv_{r+2}'=1.$$

取一与 $v_1,\cdots,v_{r+1}$ 线性无关的矢量 u, 作

$$u_{r+2}=u-\sum_{\nu=1}^{r+1}\lambda_\nu v_\nu,\quad \lambda_\nu=uFv_\nu',\quad \lambda_{r+1}=-uFv_{r+1}'\quad (\nu=1,\cdots,r)$$

则

$$u_{r+2}Fv_\nu'=0,\quad \nu=1,\cdots,r+1.$$

如果

$$u_{r+2}Fu_{r+2}'\leqslant 0,$$

则

$$\begin{pmatrix} u_1\\ u_2\\ \vdots\\ u_{r+2}\end{pmatrix} F \begin{pmatrix} u_1\\ u_2\\ \vdots\\ u_{r+2}\end{pmatrix}'=[1,\cdots,1,-1,-1].$$

由 4.4 节可知这是不可能的. 取

$$v_{r+2}=u_{r+2}/\sqrt{u_{r+2}Fu_{r+2}'},$$

即得所证.

(2) A 不能有绝对值 $\neq 1$ 的复特征根.

如果 $\rho e^{i\theta}(\rho\neq 1,\theta\neq 0$ 或 $\pi)$ 是一特征根, 而 z 是对应的特征矢量, 则

$$zA=\rho e^{i\theta}z.$$

由于

$$zF\bar{z}'=zAFA'\bar{z}'=\rho^2 zF\bar{z}'.$$

及 $\rho\neq 1$, 所以

$$zF\bar{z}'=0.$$

记 $z=x+yi$, 则得

$$xFx'+yFy'=0,\quad xFy'=0. \tag{3}$$

又由于

$$A=FA'^{-1}F,$$

所以 $\frac{1}{\rho}e^{-i\theta}$ 也是特征根, 它的特征矢量是 $w = u + iv$, 即

$$wA = \frac{1}{\rho e^{i\theta}} w,$$

同样有

$$uFu' + vFv' = 0, \quad uFv' = 0. \tag{4}$$

再由

$$zF\bar{w}' = zAFA'\bar{w}' = e^{2i\theta} zF\bar{w}',$$

所以

$$zF\bar{w}' = 0,$$

即

$$xFu' + yFv' = 0, \quad xFv' - yFu' = 0. \tag{5}$$

总之, 得到

$$\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} F \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}' = \begin{pmatrix} a & 0 & c & d \\ 0 & -a & d & -c \\ c & d & b & 0 \\ d & -c & 0 & -b \end{pmatrix}.$$

如果 $a \neq 0$, 则

$$\begin{pmatrix} I & 0 \\ -\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}^{-1} & I \end{pmatrix} \begin{pmatrix} a & 0 & c & d \\ 0 & -a & d & -c \\ c & d & b & 0 \\ d & -c & 0 & -b \end{pmatrix}$$

$$\times \begin{pmatrix} I & 0 \\ -\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}^{-1} & I \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & p & * \\ 0 & 0 & * & -p \end{pmatrix},$$

$$p = b - \frac{c^2 - d^2}{a},$$

它是一个有两个负号的对称方阵. 当 $a = 0$ 时, 也显然可见它不是仅有一个负号或奇异的方阵. 由 4.4 节的结果知道这是不可能的.

(3) 如果 A 有一实特征根 $\lambda \neq \pm 1$, 由于 $A'^{-1} = FAF$, 所以 $\frac{1}{\lambda}$ 也是一个特征根, 各对应一个特征矢量 x 与 y, 即

$$xA = \lambda x, \quad yA = \frac{1}{\lambda} y.$$

由

$$xFx' = xAFA'x' = \lambda^2 xFx',$$

可知

$$xFx' = 0,$$

同法

$$yFy' = 0.$$

命 $xFy' = a$, 则

$$\begin{pmatrix} x \\ y \end{pmatrix} F \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

若 $a = 0$, 由 4.4 节知其不可能. 取 $\dfrac{x}{a}$ 代为新的 x, 则得

$$xFx' = yFy' = 0, \quad xFy' = 1.$$

由于

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

命

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

则

$$\begin{aligned} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} A &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} A \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned}$$

$$=\begin{pmatrix} \dfrac{\lambda^{-1}+\lambda}{2} & \dfrac{\lambda^{-1}-\lambda}{2} \\ \dfrac{\lambda^{-1}-\lambda}{2} & \dfrac{\lambda^{-1}+\lambda}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

命 $\frac{1}{2}(\lambda+\lambda^{-1})=\cos h\psi$, 则

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} A=\begin{pmatrix} \cos h\psi & \sin h\psi \\ \sin h\psi & \cos h\psi \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

及

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} F \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

存在一 Lorentz 方阵 T, 以 v_1、v_2 为其最后二行, 如此则得

$$TAT^{-1}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_{m-2,1} & a_{m-2,2} & \cdots & a_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cos h\psi & \sin h\psi \\ 0 & 0 & \cdots & 0 & \sin h\psi & \cos h\psi \end{pmatrix},$$

这里 $(a_{ij})_{1\leqslant i,j\leqslant m-2}$ 是一正交方阵.

(4) 如果 A 有一绝对值等于 1 的复根, 则与 4.3 节同法可以证明存在一个 Lorentz 方阵 T 使

$$TAT^{-1}=\begin{pmatrix} \cos\theta & \sin\theta & 0 & \cdots & 0 \\ -\sin\theta & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{m3} & \cdots & a_{mm} \end{pmatrix},$$

这里 $(a_{ij})_{3\leqslant i,j\leqslant m}$ 是一个 $m-2$ 行列的 Lorentz 方阵.

4.6　续

由上节的结果可知: 所留待研究的情况是 A 仅以 $+1$ 与 -1 为特征根的情况. 假定 1 是 A 的特征根, 而 x 是其对应的特征矢量, 即

$$xA=x. \tag{1}$$

(1) 假定有一个适合于 (1) 的矢量使

$$xFx' > 0.$$

不妨假定 $xFx' = 1$. 取以 x 为第一行的 Lorentz 方阵 T, 则

$$TAT^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix},$$

这里 A_1 是 $m-1$ 行列的 Lorentz 方阵.

(2) 假定有一个适合于 (1) 的矢量使

$$xFx' < 0.$$

不妨假定 $xFx' = -1$, 取一个以 x 为末行的 Lorentz 方阵 T, 则

$$TAT^{-1} = \begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix},$$

这里 A_2 是一个 $m-1$ 行列的正交方阵.

(3) 主要难点在于处理所有使 $xA = x$ 的矢量 x 都使 $xFx' = 0$ 的情况.

如果有两个线性无关的 x, y 使

$$\begin{aligned} xA = x, \quad xFx' = 0, \\ yA = y, \quad yFy' = 0. \end{aligned} \tag{2}$$

由于

$$(\lambda x + \mu y)A = \lambda x + \mu y,$$

所以也有

$$(\lambda x + \mu y)F(\lambda x + \mu y)' = 0,$$

即得

$$xFy' = 0.$$

而

$$\begin{pmatrix} x \\ y \end{pmatrix} F \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

由 4.4 节的结果知其不可能.

因此只有一个矢量 (相差一常数倍)x 使 $xA = x$. $xFx' = 0$.

(4) 如果 A 有一个特征根 -1, 假定

$$zA = -z,$$

如果 $zFz' \neq 0$, 则用 (1), (2) 的同法解决问题, 如果 $zFz' = 0$, 则

$$xFz' = xAFA'z' = -xFz',$$

即 $xFz' = 0$. 因而

$$\begin{pmatrix} x \\ z \end{pmatrix} F \begin{pmatrix} x \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

这是不可能的.

(5) 由 (3) 及 (4) 可知现在仅需研究的是 A 仅以 $+1$(或 -1) 为特征根, 而且只有一个线性无关的矢量以 1 为特征根的 (也就是它的 Jordan 标准形是

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

的). 当 $m \geqslant 2$, 还有矢量 y 使

$$yA = y + x. \tag{3}$$

由

$$yFy' = yAFA'y' = (x+y)F(x+y)' = yFy' + 2xFy',$$

即得

$$xFy' = 0. \tag{4}$$

由于

$$\begin{pmatrix} x \\ y \end{pmatrix} F \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & yFy' \end{pmatrix}, \tag{5}$$

因此 $yFy' = a \neq 0$. 如果 $m = 2$, 则 (5) 的左边非奇异的, 因而也不可能, 所以 $m \geqslant 3$.

即还有一矢量 z 使

$$zA = z + y. \tag{6}$$

由

$$\begin{aligned} zFy' =& zAFA'y' = (z+y)F(y+x)' \\ =& zFy' + yFy' + zFx', \end{aligned}$$

可得

$$xFz' = -yFy' = -a, \tag{7}$$

又由

$$\begin{aligned} zFz' =& zAFA'z' = (z+y)F(z+y)' \\ =& zFz' + 2yFz' + yFy', \end{aligned}$$

可得

$$yFz' = -\frac{1}{2}yFy' = -\frac{1}{2}a.$$

因此得出

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} F \begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 0 & -a \\ 0 & a & -\frac{1}{2}a \\ -a & -\frac{1}{2}a & b \end{pmatrix}, \quad b = zFz'. \tag{8}$$

如果还有矢量 w 使

$$wA = w + z, \tag{9}$$

则由

$$xFw' = xAFA'w' = xF(w+z)' = xFw' + xFz',$$

即 $xFz' = 0$. 这与 $a \neq 0$ 相矛盾. 因此 $m = 3$, 而

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{10}$$

(6) 我们现在考虑对称方阵 (8), 它的行列式等于 $-a^3 (\neq 0)$, 它是 $+1, +1, -1$ 型的二次型, 因此 $a > 0$. 命

$$\begin{aligned} x^* &= \frac{1}{\sqrt{a}}x, \quad y^* = \frac{1}{\sqrt{a}}y, \\ z^* &= \frac{1}{\sqrt{a}}(\lambda x + z), \quad \lambda = \frac{b}{2a}. \end{aligned}$$

则

$$\begin{aligned} & x^*Fx^{*'} = 0, \quad x^*Fy^{*'} = 0, \quad x^*Fz^{*'} = -1, \\ & y^*Fy^* = 1, \quad y^*Fz^* = -\frac{1}{2} \end{aligned}$$

及

$$z^*Fz^{*'} = \frac{1}{a}(2\lambda xFz' + zFz') = \frac{1}{a}(-2a\lambda + b) = 0.$$

另一方面依然有

$$\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}.$$

取消 * 号, 不妨假定, 我们原来的 $a=1, b=0$. 总之, 我们有

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} F \begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 0 \end{pmatrix} \tag{11}$$

及

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

不难直接验算: 命

$$P = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & -2 \\ 0 & -\sqrt{2} & -1 \\ -\frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{8} & -\frac{1}{4} \end{pmatrix},$$

则

$$P \begin{pmatrix} 0 & 3 & 2\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 2\sqrt{2} & 3 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

及

$$P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} P' = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 0 \end{pmatrix}.$$

命

$$P^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T,$$

则

$$\begin{aligned} TA &= P^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} A = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 0 & 3 & 2\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 2\sqrt{2} & 3 \end{pmatrix} T, \end{aligned}$$

而且

$$TFT' = P^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} F \begin{pmatrix} x \\ y \\ z \end{pmatrix}' P'^{-1}$$

$$= P^{-1}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 0 \end{pmatrix} P'^{-1} = F,$$

即得所证 (同法处理特征根等于 -1 的情况).

4.7 Lorentz 相似的标准型

总之, 任何一个 Lorentz 方阵一定相似于以下四种形式之一:

(a) $$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} \cos\theta_\nu, & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix}$$
$$\dot{+} \underbrace{1 \dot{+} \cdots \dot{+} 1}_{i个} \dot{+} \underbrace{(-1) \dot{+} \cdots \dot{+} (-1)}_{s个} \dot{+} (\pm 1),$$

(b) $$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} \cos\theta_\nu, & \sin\theta_\nu \\ -\sin\theta_\nu, & \cos\theta_\nu \end{pmatrix}$$
$$\dot{+} \underbrace{1 \dot{+} \cdots \dot{+} 1}_{i个} \dot{+} \underbrace{(-1) \dot{+} \cdots \dot{+}}_{s个} (-1)$$
$$\dot{+} \begin{pmatrix} 0 & 3 & 2\sqrt{2} \\ -1 & 0 & 0 \\ 0 & 2\sqrt{2} & 3 \end{pmatrix}.$$

(c) $$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix}$$
$$\dot{+} \underbrace{1 \dot{+} \cdots \dot{+} 1}_{i个} \dot{+} \underbrace{(-1) \dot{+} \cdots \dot{+} (-1)}_{s个}$$
$$\dot{+} \begin{pmatrix} 0 & -3 & -2\sqrt{2} \\ 1 & 0 & 0 \\ 0 & -2\sqrt{2} & -3 \end{pmatrix}.$$

$$(\mathrm{d})\quad \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix}$$

$$\dot{+} \underbrace{1 \dot{+} \cdots \dot{+} 1}_{i\text{个}} \dot{+} \underbrace{(-1) \dot{+} \cdots \dot{+} (-1)}_{s\text{个}}$$

$$\dot{+} \begin{pmatrix} \cos h\psi & \sin h\psi \\ \sin h\psi & \cos h\psi \end{pmatrix}.$$

注意, 与正交方阵不同之处在于: 正交方阵的相似性可以完全由特征方阵来决定, 但 Lorentz 方阵则不能, 不但不能由特征方阵来决定, 而且不能由初等因子来决定, 除去初等因子这外, 还必须看其右下角元素的符号, 即不难证明: Lorentz 方阵 Lorentz 相似的必要且充分条件是它们有相同的初等因子, 而且右下角的元素同号.

也不难证明 Lorentz 方阵对应于非 ± 1 的特征根的初等因子都是单的, 而等于 ± 1 时也仅有一次, 三次两种可能性.

4.8 对合变换

定义　一个 $n+2$ 阶的 Lorentz 方阵 A, 若适合

$$A^2 = \rho I, \quad \rho = \bar{\rho} \neq 0, \tag{1}$$

则称为对合.

由于 $\det A^2 = 1$, 所以 $\rho^{n+2} = 1$, 因此 $\rho = 1$.

从上节知对合 Lorentz 方阵 Lorentz 相似于标准型

$$[1, \cdots, 1, -1, \cdots, -1 \pm 1], \tag{2}$$

所以下面两类对合 Lorentz 方阵有其基本的重要性:

(i) Lorentz 相似于

$$[-1, 1, \cdots, 1, 1, 1] \tag{3}$$

的对合 Lorentz 方阵, 称为照镜、空间对称;

(ii) Lorentz 相似于

$$[1, \cdots, 1, 1, -1] \tag{4}$$

的对合 Lorentz 方阵, 称为反演、时间对称.

其基本重要性在于其他的对合都是由有限个相互可交换的属于 (i), (ii) 的对合相乘而得到, 同时形如 (i) 和 (ii) 的对合互不 Lorentz 相似.

现在来定出照镜和反演的一般形式:

(i) 由于对任一照镜 A, 存在 Lorentz 方阵 P, 使得

$$PAP^{-1}=[-1,1,\cdots,1,1,1]=I-2[1,0,\cdots,0].$$

所以

$$A=I-2P^{-1}[1,0,\cdots,0]P.$$

然而由 $P\in L_{(n+1,1)}$ 可知 $P^{-1}=FP'F$, 代入有

$$\begin{aligned}A=&I-2[1,\cdots,1,-1]P'[1,\cdots,1,-1][1,0,\cdots,0]P\\ =&I-2[1,\cdots,1,-1]p'p,\end{aligned}$$

其中 p 是 P 的第一个行向量, 由于 $P\in L_{(n+1,1)}$, 故 $pFp'=1$, 所以照镜的一般形式为

$$A=I-2Fp'p,\quad pFp'=1. \tag{5}$$

(ii) 由于对任一反演 A, 存在 Lorentz 方阵 P, 使得

$$PAP'=[1,\cdots,1,-1]=I-2[0,\cdots,0,1],$$

所以

$$A=I-2P^{-1}[0,\cdots,0,1]P,$$

由于 $P^{-1}=FP'F$, 代入可知

$$A=I+2FP'[0,\cdots,0,1]P=I+2Fq'q,$$

其中 q 是 P 的 $n+2$ 个行向量, 由于 $P\in L_{(n+1,1)}$, 故 $qFq'=-1$, 所以反演的一般形式为

$$A=I+2Fq'q,\quad qFq'=-1. \tag{6}$$

对任一对合 Lorentz 方阵 A 利用 (4.1.3) 得到方阵

$$M=\left[I^{(n)},\begin{pmatrix}1&-1\\1&1\end{pmatrix}\right]^{-1}A\left[I^{(n)},\begin{pmatrix}1&-1\\1&1\end{pmatrix}\right]. \tag{7}$$

我们称变形

$$(y,yy',1)=\rho(x,xx',1)M \tag{8}$$

为对合. 当 A 是照镜时, 对合 (8) 称为照镜, 当 A 是反演时, 对合 (8) 称为反演.

照镜的标准型为

$$y_1=-x_1,y_2=x_2,\cdots,y_n=x_n. \tag{9}$$

反演的标准型为

$$y = -\frac{x}{xx'}. \tag{10}$$

前者由于

$$A = [-1, 1, \cdots, 1],$$

故

$$M = [-1, 1, \cdots, 1],$$

后者由于

$$A = [1, \cdots, 1, -1],$$

故

$$M = \left[1, \cdots, 1, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right].$$

所以

$$y = \rho x, \quad yy' = -\rho, \quad -\rho xx' = 1,$$

即

$$\rho = -\frac{1}{xx'},$$

所以

$$y = -\frac{x}{xx'}.$$

第 5 讲　球几何的基本定理——兼论狭义相对论的基本定理

5.1　引　　言

1946 年作者研究矩阵几何学时, 用了一个方法. 这方法可以用来处理 n 维球空间的基本定理, 也就是用球相切性可以推出球几何学的基本定理, 也就是不必用变换的解析性, 甚至连续性, 就可以推导出其变换群是球几何学的 Lie 群、Laguerre 群.

这里只讲三维空间的球几何学, 其原因是一方面比较具体, 易于接受, 另一方面企图使这一成果能为一些物理工作者所注意. 实质上三维的球几何学就是狭义相对论的另一表达形式, 而这点往往未被认识. 例如, 1961 年, B.A.Фок 写的 Теория пространства, времена и тяготения一书 (有中译本, 1965 年科学出版社出版), 书中仍旧用的是 Riemann 几何、解析群论的老路子. 而中、英、德等文的译本中也都未注意到这一点, 而加以应有的注记.

对狭义相对论来说, 原有两个假设:

(A) 相对性原理中要求匀速直线运动还是匀速直线运动.

(B) 光速不变原理是假设光以常速 c 作直线运动.

我们现在处理的方法是有了光速不变原理, 就可以推出 Lorentz 群了, 就是相对性原理中要求匀速直线运动还是匀速直线运动是推论而不是假设. 这给我们提供了方便, 如果要验证或推翻上述二点, 只要用实验来检验光速不变性就够了. 至于如何推到 n 维球几何学及一般 Hermite 方阵几何学上去, 就更不是我们的着重点了.

我们讲的球是以 (x, y, z) 为球心, R 为半径的球. 如果 R 是正的, 则球带一个向外的箭头 (图 1), 如 R 是负的, 则球带一向内的箭头 (图 2).

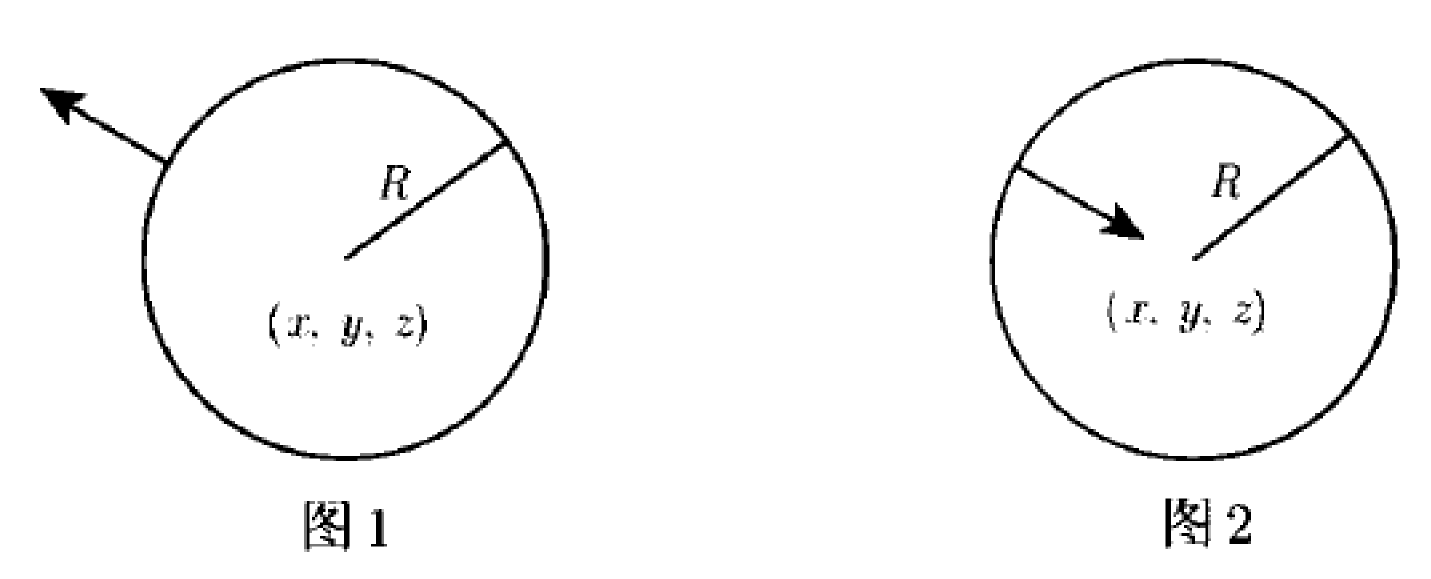


图 1　　　　图 2

两球相切的条件是它们所带的箭头在切点须同向.

两球 (x,y,z,R), (x_1, y_1, z_1, R_1) 相切的条件是

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=(R-R_1)^2. \tag{1}$$

当 R, R_1 是同号时是内切, 异号时是外切, 即如图 3. 我们的球几何学就是以球为元素所形成空间的几何学.

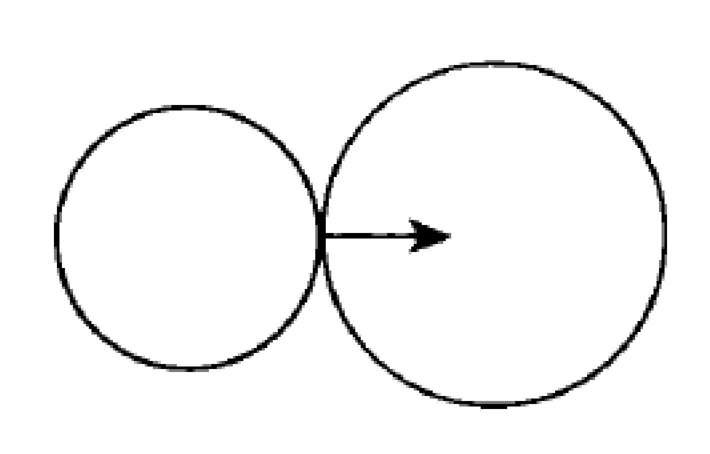
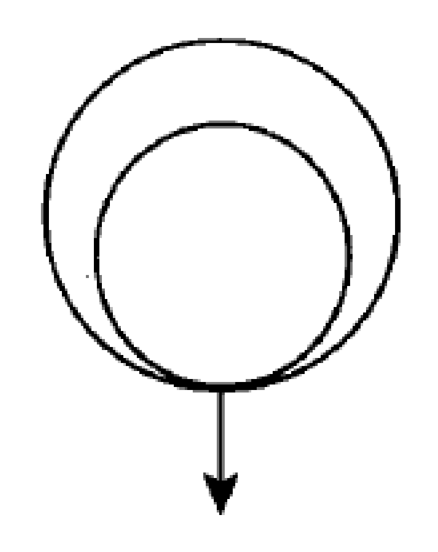



图3

用二行二列的 Hermite 方阵

$$H=\begin{pmatrix} R+x & y+iz \\ y-iz & R-x \end{pmatrix}$$

表示 (x,y,z) 为中心, R 为半径的球, 则相切条件 (1) 就是

$$H-H_1=\begin{pmatrix} R-R_1+x-x_1, & y-y_1+i(z-z_1) \\ y-y_1-i(z-z_1), & R-R_1-(x-x_1) \end{pmatrix}$$

的行列式为 0. 相切就是我们矩阵几何里称为的粘切. 我们的问题是找出使二行二列 Hermite 方阵一一变为 Hermite 方阵保持粘切关系不变的最大的群来.

改为相对论的语言, 在地点为 (x,y,z), 时间为 t 的时空点, 用 Hermite 方阵

$$\begin{pmatrix} ct+x, & y+iz \\ y-iz, & ct-x \end{pmatrix}$$

来表示. 粘切的条件就是两点的距离正好等于光行的时间乘光速.

不像 Фок 那样, 在这里仅需要二行二列方阵运算就够了.

5.2　匀速直线运动

一件事的发生必有时间 t, 必有地点 (x,y,z), 匀速直线运动可以表成为

$$x-x_0=v_x(t-t_0),\quad y-y_0=v_y(t-t_0),\quad z-z_0=v_z(t-t_0). \tag{1}$$

这是沿方向 $v_x : v_y : v_z$, 依速度 $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ 前进的直线匀速运动, 而当时间 $t = t_0$ 时, 通过 (x_0, y_0, z_0).

将 (1) 改写成为

$$\begin{cases} x - x_0 = \alpha\tau, \\ y - y_0 = \beta\tau, \\ z - z_0 = \gamma\tau, \\ t - t_0 = \delta\tau. \end{cases} \quad -\infty < \tau < \infty. \tag{2}$$

由此可以看到三维空间的匀速直线运动 (1) 与四维空间的直线 (2) 一一对应.

由四维空间的仿射几何的基本定理 (参阅华罗庚、万哲先《典型群》. 上海科技出版社, 1962), 可以知道把四维空间一一地变为其自己, 把直线变为直线的变形一定是仿射变换.

如果引进无穷远点使仿射空间扩充成为射影空间, 则由射影几何的基本定理 (参阅《典型群》) 可知所定出的变换就是射影变换. 这就是 Фок 书中附录一的第一个结论. 这里我们不但没有要求变换有三阶偏微商的条件, 连连续性都未假定.

5.3 Hermite 方阵的几何学

以下如不特殊声明, 大写字母 $A, B, C, \cdots$ 表二行二列的复数方阵. 适合于 $P = \overline{P'}$ 的方阵称为 Hermite 方阵或简称 H 方阵. 显然变换

$$X^* = AX\overline{A'} + X_0, \qquad \overline{X_0'} = X_0 \tag{1}$$

把 Hermite 方阵 X 变为 Hermite 方阵 X^*, 这里 A 是可逆方阵. 除此之外, 还有变换

$$X^* = -X, \tag{2}$$

$$X^* = X', \tag{3}$$

两个 H 方阵 X_1, X_2, 如果 $X_1 - X_2$ 之秩等于 1, 则 X_1, X_2, 称为粘切. 这样定义是为了更方便地把这章的结果推到 n 行列的 H 方阵, 现在实际上就是

$$|X_1 - X_2| = 0. \tag{4}$$

我们基本定理就是: 把 H 方阵变为 H 方阵的一一对应, 且保持粘切关系不变的变形, 就是由 (1), (2), (3) 所演成的群.

我们现在来说明 (1), (2), (3) 所成的群与 Lorentz 群的关系, 对应一个 H 方阵

$$X = \begin{pmatrix} ct+x & y+iz \\ y-iz & ct-x \end{pmatrix}, \tag{5}$$

有一个矢量

$$\omega = (ct, x, y, z). \tag{6}$$

每一个 H 方阵 X 的变换对应于一个矢量 ω 的变换. 变换

$$X^* = X + X_0 \tag{7}$$

就对应于平移

$$\omega^* = \omega + \omega_0. \tag{8}$$

因此只要考虑

$$X^* = AX\overline{A'} \tag{9}$$

及 (2), (3). 它们所对应的线性变换就是

$$\omega^* = \omega L(A). \tag{10}$$

对 (9) 取行列式得

$$\omega^*[1, -1, -1, -1]\omega^{*\prime} = \omega[1, -1, -1, -1]\omega'|A\overline{A'}|,$$

即

$$\omega L[1, -1, -1, -1]L'\omega' = \omega[1, -1, -1, -1]\omega'|A\overline{A'}|.$$

因此

$$L[1, -1, -1, -1]L' = |A\overline{A'}|[1, -1, -1, -1], \tag{11}$$

即 $L(A)/|A\overline{A'}|^{\frac{1}{2}}$ 是 Lorentz 变换. 但须注意 A 与 $e^{i\theta}A$ 代表同一 Lorentz 变换. 因此不妨假定

$$|A| = \rho^2 > 0.$$

当 $A = \rho I$ 时, 所对应的变换是度量变换. 因此, $A = \rho B$, 而 $|B| = 1$. 而两个 A, 即 $\pm A$, 对应于同一个 Lorentz 变换及同一个度量变换. 以下假设 $|A| = 1$.

现在先研究一些特殊的 A 与 $L(A)$ 的对应关系:

(i)

$$A=\begin{pmatrix}\cos\frac{1}{2}\theta, & -\sin\frac{1}{2}\theta\\ \sin\frac{1}{2}\theta, & \cos\frac{1}{2}\theta\end{pmatrix},$$

$$L(A)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

即绕 z 轴的旋转.

(ii)

$$A=\begin{pmatrix}e^{\frac{1}{2}i\theta} & 0\\ 0 & e^{-\frac{1}{2}i\theta}\end{pmatrix},$$

$$L(A)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\theta & \sin\theta\\ 0 & 0 & -\sin\theta & \cos\theta\end{pmatrix},$$

即绕 x 轴的旋转.

(iii)

$$A=\begin{pmatrix}\cos\frac{1}{2}\theta, & -i\sin\frac{1}{2}\theta\\ -i\sin\frac{1}{2}\theta, & \cos\frac{1}{2}\theta\end{pmatrix},$$

$$L(A)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & 0 & \sin\theta\\ 0 & 0 & 1 & 0\\ 0 & -\sin\theta & 0 & \cos\theta\end{pmatrix},$$

即绕 y 轴的旋转.

(iv)

$$A=\begin{pmatrix} e^{\frac{1}{2}\psi} & 0 \\ 0 & e^{-\frac{1}{2}\psi} \end{pmatrix} \quad (\psi>0),$$

$$L(A)=\begin{pmatrix} \cosh\psi & \sinh\psi & 0 & 0 \\ \sinh\psi & \cosh\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

这就是双曲旋转.

(v) $X^*=X', t^*=t, x^*=x, y^*=y, z^*=-z$, 即是空间反演.

(vi) $X^*=-AX'\overline{A'}, \quad A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;

$$t^*=-t, \quad x^*=x, \quad y^*=y, \quad z^*=z,$$

即是时间反演.

由于 (i), (ii), (iii), $L(A)$ 演出所有的旋转群, 而 A 演出所有行列式为 1 的酉群, 因此行列式为 1 的酉群的 $\pm U$ 与旋转群对应. (i), (ii), (iii), (v) 的 $L(A)$ 给出正交群. 而 (i), (ii), (iii), (iv) 的 $L(A)$ 给出 L_+^+ 的所有变换. (i), (ii), (iii), (vi), (v), (vi) 的 $L(A)$ 给出 Lorentz 群的所有的变换.

这阐明了我们所用的符号与狭义相对论的符号的一致性. 即用我们符号所得出的变换是 Lorentz 变换, 而且是全部的 Lorentz 变换. 值得一提的是由 (i), (ii), (iii) 中的 A 演出的是二维特殊 U_2 群, 即适合 $U\overline{U'}=I, |U|=1$ 的 U 所成的群. 因此可见 $\pm U_2$ 与旋转群 Γ_3 是对应的.

附记 对任一 $A(|A|=1)$, 一定有一方阵 $P(|P|=1)$ 使

$$PAP^{-1}=\begin{pmatrix} e^{\frac{1}{2}\psi} & 0 \\ 0 & e^{-\frac{1}{2}\psi} \end{pmatrix}, \quad \begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

各得出在 Lorentz 群下, Lorentz 方阵的标准型除双曲变换及旋转外还有一种, 这就是 4.5 节的一个奇特的标准型. 但用 2×2 方阵就特别显著了, 就是 A 有重特征根而且初等因子非单纯的. 致于这样的变换的相对论意义这是作者至今不能了解的.

我们的基本定理的物理意义是 "光以常速 c 作直线运动" 刻画了 Einstein 的狭义相对论. 实际上如果我们假定消息传递有上界, 我们这一套数学工具还是可用的. 今后用 x_0, x_1, x_2, x_3 来替代 t, x, y, z.

5.4 三维空间中使单位球不变的仿射变换

引理 1 一个使单位球不变的仿射变换一定是正交变换 (参阅 3.1 节).

证 假定

$$y = xA + b \tag{1}$$

是一个这样的变换, 这里 x, y, b 是三维矢量, A 是三行三列的非奇异方阵, (1) 把单位球 $xx' = 1$ 变为 $yy' = 1$.

可知有两个正交方阵 Γ_1, Γ_2 使

$$\Gamma_1 A \Gamma_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_r > 0.$$

因此不失普遍性可以假定

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

特别取单位球上的点

$$x = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, 0\right),$$

则得

$$1 = y_1^2 + y_2^2 + y_3^2 = \left(\lambda_1 \frac{1-t^2}{1+t^2} + b_1\right)^2 + \left(\frac{2t\lambda_2}{1+t^2} + b_2\right)^2 + b_3^2,$$

即

$$(1+t^2)^2 = (\lambda_1(1-t^2) + b_1(1+t^2))^2 + (2t\lambda_2 + b_2(1+t^2))^2 + b_3^2(1+t^2)^2.$$

比较 t 的系数, 得 $\lambda_2 b_2 = 0$, 即得 $b_2 = 0$. 同法证明 $b_1 = b_3 = 0$. 代入后再比较 t^4 的系数, 得 $\lambda_1^2 = 1$, 即得 $\lambda_1 = 1$, 同法得 $\lambda_2 = \lambda_3 = 1$, 即 (1) 是一恒等变换.

为了将来引用方便, 我们把三维空间及刚体运动群写成为二行二列的方阵形式:

$$x = (x_1, x_2, x_3) \Longleftrightarrow X = \begin{pmatrix} x_1, & x_2 + ix_3 \\ x_2 - ix_3, & -x_1 \end{pmatrix}.$$

即三维空间的一点 (x_1, x_2, x_3) 可用二行二列 t 为 0 的 H 方阵表出来.

直线

$$x = a + \lambda b \Longleftrightarrow X = A + \lambda B \quad (-\infty < \lambda < \infty).$$

平面

$$ax' = \mu \Longleftrightarrow \mathrm{tr}(AX) = 2\mu.$$

两点的距离的平方

$$(x-u)(x-u)' \Longleftrightarrow -|X-U|.$$

由上一节可知, 刚体运动群

$$\left.\begin{array}{l} y = x\Gamma + b \\ \Gamma\Gamma' = I, |\Gamma| = 1 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} Y = UX\overline{U}' + B, & \overline{B}' = B, \\ U\overline{U}' = I, & |U| = 1. \end{array}\right.$$

注意右边 U 与 $-U$ 代表同一变换.

欧几里得群是刚体运动群再添上一反射

$$y_1 = x_1, y_2 = x_2, y_3 = -x_3 \Longleftrightarrow Y = X'.$$

所以欧几里得群可以用

$$Y = UX\overline{U}' + B$$

添加 $Y = X'$ 来表达出来.

5.5 粘切子空间

我们现在回来考虑 5.3 节中所提出的问题. 我们所讨论的空间是四维时空空间, 也就是所有的二行二列的 H 方阵所成的空间, 这空间的点就是指一二行二列的 H 方阵. 变换指 5.3 节中所定义的仿射变换. 我们将用粘切关系来定义一些几何图形.

定义 1 命 A, B 是两个粘切的点. 与 A, B 都粘切的点所成的集合称为粘切子空间.

定理 1 任一粘切子空间可以变为标准型

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad -\infty < a < \infty. \tag{1}$$

也就是说, 在仿射变换下, 粘切子空间成一可递集合.

证 经仿射变换, 不妨假定

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

命

$$X=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \gamma\end{pmatrix},\qquad \alpha,\gamma \text{ 实数}.$$

由粘切关系

$$|A-X|=|B-X|=0,$$

可得

$$\alpha\gamma-|\beta|^2=0,\qquad (\alpha-1)\gamma-|\beta|^2=0.$$

所以 $\gamma=0,\beta=0$. 即得所证.

定理 2 两个不同的粘切子空间 Σ_1, Σ_2 最多只能有一个公共点. 如果有一个公共点, 它们可同时变成为

$$\Sigma_1:\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix},\quad -\infty<a<\infty,$$

$$\Sigma_2:\begin{pmatrix}0 & 0\\ 0 & b\end{pmatrix},\quad -\infty<b<\infty.$$

证 由定义, 不同的粘切子空间不可能有两个公共点.

假定公共点是 0, 并不妨假定其一已经是标准型 Σ_1, Σ_2 的元素的形成一定是

$$d\begin{pmatrix}|s|^2 & s\bar{t}\\ \bar{s}t & |t|^2\end{pmatrix},\qquad t\neq 0, d\neq 0. \tag{2}$$

变换

$$Y=\begin{pmatrix}1 & -st^{-1}\\ 0 & 1\end{pmatrix}X\overline{\begin{pmatrix}1 & -st^{-1}\\ 0 & 1\end{pmatrix}}'$$

使 Σ_1 不变, 但把 (2) 变为

$$d\begin{pmatrix}1 & -st^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}s\bar{s} & s\bar{t}\\ \bar{s}t & t\bar{t}\end{pmatrix}\overline{\begin{pmatrix}1 & -st^{-1}\\ 0 & 1\end{pmatrix}}'=d\begin{pmatrix}0 & 0\\ 0 & |t|^2\end{pmatrix},$$

这和 0 获得 Σ_2.

定理 3 仅有一公共点的三个粘切子空间可以同时变为

$$\Sigma_1 : \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad -\infty < a < \infty,$$

$$\Sigma_2 : \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad -\infty < b < \infty,$$

$$\Sigma_3 : d\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad -\infty < d < \infty.$$

证　假定其中的两个已经变成标准型 Σ_1, Σ_2 了. 在 Σ_3 中有一元素如 (2) 形, 其 s, t 都不能为 0, 不然它就属于 Σ_1 或 Σ_2 了.

变换

$$Y = \begin{pmatrix} s^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} X \overline{\begin{pmatrix} s^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}}'$$

使 Σ_1, Σ_2 不变, 而把 (2) 变为

$$d\begin{pmatrix} s^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} |s|^2 & s\bar{t} \\ \bar{s}t & |t|^2 \end{pmatrix}\overline{\begin{pmatrix} s^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}}' = d\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

即得定理.

5.6　空相平面 (或二维空相子空间)

定义 1　命 Σ_1 与 Σ_2 是两个仅有一公共点的粘切子空间, 所有既不与 Σ_1 粘切, 又不与 Σ_2 粘切的点所成的集合定义为二维空相子空间 (或空相平面).

定理 1　空相平面是可递的, 其标准型是

$$\begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}, \tag{1}$$

这里 β 过所有的复数.

证　不妨假定 Σ_1, Σ_2 是 5.5 节定理 2 的形式. 命

$$X = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \gamma \end{pmatrix}$$

是空相平面中的一点. 由于没有 a, b 能使

$$\begin{vmatrix} \alpha - a & \beta \\ \bar{\beta} & \gamma \end{vmatrix} = 0, \qquad \begin{vmatrix} \alpha & \beta \\ \bar{\beta} & \gamma - b \end{vmatrix} = 0,$$

所以 $\gamma=\alpha=0$. 即得所证.

二维空相子空间的一般形式是

$$X=A\begin{pmatrix}0&\tau\\\bar{\tau}&0\end{pmatrix}\overline{A'}+B. \tag{2}$$

这里 τ 过所有的复数.

5.7 空相直线

定义 两个不同的空相平面如果有一个以上的交点, 则交点的集合称为一条空相直线 (或一维空相子空间).

定理 1 空相直线有以下的标准型:

$$\begin{pmatrix}0&\rho\\\rho&0\end{pmatrix},\qquad -\infty<\rho<\infty. \tag{1}$$

证 不妨假定其一空相平面就是

$$\begin{pmatrix}0&\xi\\\bar{\xi}&0\end{pmatrix},\qquad \xi \text{ 为复数}. \tag{2}$$

另一由 5.6 节 (2) 给出, 其两个公共点是 $\xi=\xi_0, \xi=\xi_1$. 仿射变换

$$Y=\begin{pmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{pmatrix}\left(X-\begin{pmatrix}0&\xi_0\\\bar{\xi}_0&0\end{pmatrix}\right)\begin{pmatrix}e^{-i\theta}&0\\0&e^{i\theta}\end{pmatrix}$$

把 ξ_0, ξ_1 变为 $\xi=0$ 及 $\xi=\rho_1$(实数)($\xi_1-\xi_0=\rho_1 e^{-2i\theta}$). 由 5.6 节 (2) 得

$$0=A\begin{pmatrix}0&\tau_0\\\bar{\tau}_0&0\end{pmatrix}\overline{A}'+B. \tag{3}$$

$$\begin{pmatrix}0&\rho_1\\\rho_1&0\end{pmatrix}=A\begin{pmatrix}0&\tau_1\\\bar{\tau}_1&0\end{pmatrix}\overline{A}'+B. \tag{4}$$

对任一实数 ρ, (3) 乘以 $\left(1-\dfrac{\rho}{\rho_1}\right)$ 加上 (4) 乘以 $\dfrac{\rho}{\rho_1}$, 得

$$\begin{pmatrix}0&\rho\\\rho&0\end{pmatrix}=A\begin{pmatrix}0&\left(1-\dfrac{\rho}{\rho_1}\right)\tau_0+\dfrac{\rho}{\rho_1}\tau_1\\\left(1-\dfrac{\rho}{\rho_1}\right)\bar{\tau}_0+\dfrac{\rho}{\rho_1}\bar{\tau}_1&0\end{pmatrix}\overline{A}'+B,$$

即可知 (1) 是两空相平面的相交部分.

再证明 (1) 之外无其他的点, 如还有一 ξ_0(非实数), 即

$$\begin{pmatrix} 0 & \xi_0 \\ \bar{\xi}_0 & 0 \end{pmatrix} = A \begin{pmatrix} 0 & \tau_2 \\ \bar{\tau}_2 & B \end{pmatrix} \overline{A'} + B, \tag{5}$$

对任一复数 ξ, 可以取实数 β, 使 $\xi - \beta\xi_0 = \alpha$ 是实数. (3) 乘以 $1 - \dfrac{\alpha}{\rho_1} - \beta$, 加上 (4) 乘以 α/ρ_1, 再加上 (5) 乘以 β, 得

$$\begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix} = A \begin{pmatrix} 0 & \left(1-\dfrac{\alpha}{\rho_1}-\beta\right)\tau_0+\dfrac{\alpha}{\rho_1}\tau_1+\beta\tau_2 \\ \overline{\left(1-\dfrac{\alpha}{\rho_1}-\beta\right)\tau_0+\dfrac{\alpha}{\rho_1}\tau_1+\beta\tau_2} & 0 \end{pmatrix} \overline{A'} + B,$$

即与空相平面 (2) 全同.

5.8 点　　对

依 $|A-B| \gtreqless 0$ 把点对 A, B 分为三个类型: 能有因果关系的点对 (或称双曲对); 粘切对 (或称抛物对); 不能有因果关系的点对 (或称椭圆对).

定理 1　空相平面上任意二点都无因果关系.

定理 2　在仿射变换下, 任一双曲对可变为标准型

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

任一抛物对可变为标准型

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

任一椭圆对可变为标准型

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

定理 2 的证明是显然的, 从略. 由定理 2 可立即推出定理 1. 由于任何两个有因果关系的点都可以同时变到 0, I. 如有同一空相平面上, 即有

$$0 = A \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix} \overline{A'} + B, \qquad I = A \begin{pmatrix} 0 & \xi_1 \\ \xi_1 & 0 \end{pmatrix} \overline{A'} + B.$$

相减取行列式得：$1=-|\xi-\xi_1|^2$, 这不可能.

定理 3 在保持粘切关系的一一对应之下, 任一点对的类型不变.

证 抛物对显然变为抛物对. 故只要证明任一椭圆对变为椭圆对即可, 不妨认为这椭圆对就是标准型 $\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. 这二点的连线是空相直线 $\begin{pmatrix}0&\rho_1\\\rho_1&0\end{pmatrix}$, $-\infty<\rho_1<\infty$. 由空相平面变为空相平面, 故空相直线变为空相直线, 而由 5.7 节定理 1 得到空相直线的一般形式是

$$A\begin{pmatrix}0&\rho\\\rho&0\end{pmatrix}\overline{A'}+B,\quad -\infty<\rho<\infty.$$

其上任意两点显然都是椭圆对, 所以椭圆对变为椭圆对. 因而, 双曲对也一定变为双曲对.

5.9 三维空相子空间

定义 1 在空相平面 S_2 外, 存在一点 P 与 S_2 上任何一点都不能有因果关系. 作 P 与 S_2 的所有点的直线 (空相), 这些线上的点的集合称为三维空相子空间.

定理 1 三维空相子空间的标准型是

$$\begin{pmatrix}x_1, & x_2+ix_3\\ x_2-ix_3, & -x_1\end{pmatrix},\qquad -\infty<x_1,x_2,x_3<\infty. \tag{1}$$

证 不妨假定空相平面 S_2 就是

$$\begin{pmatrix}0&q\\\bar q&0\end{pmatrix}. \tag{2}$$

命 $P=\begin{pmatrix}p_0&q_0\\\bar q_0&r_0\end{pmatrix}$ 是 S_2 外之一点. 变形

$$Y=X-\begin{pmatrix}0&q_0\\\bar q_0&0\end{pmatrix}$$

使 (2) 不变, 而把 P 变为

$$P=\begin{pmatrix}p_0&0\\0&r_0\end{pmatrix}.$$

由于与 0 无因果关系, 所以 $p_0r_0<0$. 再由变换

$$\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}\begin{pmatrix}p_0&0\\0&r_0\end{pmatrix}\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}.$$

可以假定

$$P=p\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{3}$$

由 (2) 与 (3) 作联线.

$$\mu p\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+(1-\mu)\begin{pmatrix}0 & q\\ \bar{q} & 0\end{pmatrix}=\begin{pmatrix}\mu p & (1-\mu)q\\ (1-\mu)\bar{q} & -\mu p\end{pmatrix},$$

即得所证.

5.10　基本定理的证明

1. 命 π_3 是一个三维空相子空间. 命 P 是 π_3 外的一点. 我们有一仿射变换把 π_3 变为

$$\begin{pmatrix}x_1 & x_2+ix_3\\ x_2-ix_3 & -x_1\end{pmatrix} \tag{1}$$

及 P 变为

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{2}$$

其证明是: 不妨先假定 π_3 已变为 (1). 命

$$P=\begin{pmatrix}p_0+p_1, & p_2+ip_3\\ p_2-ip_3, & p_0-p_1\end{pmatrix},\qquad p_0\neq 0.$$

则

$$Y=\frac{1}{p_0}\left[X-\begin{pmatrix}p_1, & p_2+ip_3\\ p_2-ip_3, & -p_1\end{pmatrix}\right]$$

把 (1) 变为自己, P 变为 (2).

2. 命

$$Y=\varPhi(X) \tag{3}$$

是一个把 H 方阵变为 H 方阵的一一对应, 而且保持粘切关系. 由三维空相子空间的定义及 5.8 节 (注意纯由粘切关系得来的). 因此我们不妨假定

$$\varPhi\begin{pmatrix}x_1 & x_2+ix_3\\ x_2-ix_3 & -x_1\end{pmatrix}=\begin{pmatrix}y_1 & y_2+iy_3\\ y_2-iy_3 & -y_1\end{pmatrix}.$$

这样的子空间用 π_3 表之, 且有

$$\varPhi(I)=I. \tag{4}$$

在 π_3 上得到一个普通的三维空间 (x_1, x_2, x_3), 不难证明, 其中的平面是二维空相平面, 直线必是一维空相子空间. 由仿射空间的基本定理, 有把直线变为直线性质的变换就是三维空间的仿射变换. 不仅如此, 由于粘切关系

$$\left|I - \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix}\right| = 1 - (x_1^2 + x_2^2 + x_3^2) = 0,$$

这仿射变换还要保持单位球不变.

由 5.3 节和 5.4 节的结果知道, 当 $X, Y \in \pi_3$ 时, 则

$$Y = UX\overline{U'}, \qquad U\overline{U'} = I.$$

3. 因此不妨假定

$$\Phi\begin{pmatrix} \xi_1, & \xi_2 + i\xi_3 \\ \xi_2 - i\xi_3, & -\xi_1 \end{pmatrix} = \begin{pmatrix} \xi_1, & \xi_2 + i\xi_3 \\ \xi_2 - i\xi_3, & -\xi_1 \end{pmatrix}. \tag{5}$$

一般的点

$$\begin{pmatrix} x_0 + x_1, & x_2 + ix_3 \\ x_2 - ix_3, & x_0 - x_1 \end{pmatrix}, \qquad x_0 \neq 0$$

与 π_3 中的点粘切的条件是

$$x_0^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2. \tag{6}$$

由 (3) 可知凡适合于 (6) 的 ξ_1, ξ_2, ξ_3 也一定适合于

$$y_0^2 = (y_1 - \xi_1)^2 + (y_2 - \xi_2)^2 + (y_3 - \xi_3)^2; \tag{7}$$

反之亦真.

π_3 中有两点 $\xi_1 = x_1 \pm x_0, \xi_2 = x_2, \xi_3 = x_3$ 适合 (6), 代入 (7), 得到

$$y_0^2 = (y_1 - x_1 \pm x_0)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2. \tag{8}$$

由于对 $\pm$ 号都对, 所以

$$(y_1 - x_1)x_0 = 0,$$

即得 $y_1 = x_1$. 同法证明 $x_2 = y_2$, $x_3 = y_3$ 及 $x_0 = \pm y_0$. 因此得出结论

$$\Phi\begin{pmatrix} x_0 + x_1, & x_2 + ix_3 \\ x_2 - ix_3, & x_0 - x_1 \end{pmatrix} = \begin{pmatrix} \pm x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & \pm x_0 - x_1 \end{pmatrix}, \tag{9}$$

离恒等变换只差一 $\pm$ 号了.

4. 我们已经知道 $\Phi(I)=I$. 我们再证 Φ 也使

$$\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\dfrac{\lambda}{2}+\dfrac{\lambda}{2} & 0 \\ 0 & 1+\dfrac{\lambda}{2}-\dfrac{\lambda}{2} \end{pmatrix} \tag{10}$$

不变. 由 (9) 可知如果它变, 只能变为

$$\begin{pmatrix} -\left(1+\dfrac{\lambda}{2}\right)+\dfrac{\lambda}{2} & 0 \\ 0 & -\left(1+\dfrac{\lambda}{2}\right)-\dfrac{\lambda}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1-\lambda \end{pmatrix}.$$

它不和 I 粘切. 同法, Φ 也使

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+\mu \end{pmatrix} \tag{11}$$

不变.

再证, Φ 也使

$$\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1+\mu \end{pmatrix} = \begin{pmatrix} 1+\dfrac{1}{2}(\lambda+\mu)+\dfrac{1}{2}(\lambda-\mu) & 0 \\ 0 & 1+\dfrac{1}{2}(\lambda+\mu)-\dfrac{1}{2}(\lambda-\mu) \end{pmatrix} \tag{12}$$

不变, 若不然由 (9) 它只能变到

$$\begin{pmatrix} -1-\dfrac{1}{2}(\lambda+\mu)+\dfrac{1}{2}(\lambda-\mu) & 0 \\ 0 & -1-\dfrac{1}{2}(\lambda+\mu)-\dfrac{1}{2}(\lambda-\mu) \end{pmatrix}$$
$$= \begin{pmatrix} -1-\mu & 0 \\ 0 & -1-\lambda \end{pmatrix},$$

与 (10) 及 (11) 粘切的可能性是

$$(2+\lambda+\mu)(2+\lambda)=0, \quad (2+\lambda+\mu)(2+\mu)=0.$$

即仅有 $\lambda=\mu=-2$, 及 $\lambda+\mu=-2$ 两个例外, 即

$$\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}, \quad \begin{pmatrix} 1+\lambda & 0 \\ 0 & -(1+\lambda) \end{pmatrix}.$$

后者本来不变, 前者不变为其自己, 则变为 I, 这不可能. 因此也是不变的.

5. 现在已知形如

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} y_0+y_1 & 0 \\ 0 & y_0-y_1 \end{pmatrix}$$

的方阵不变. 考虑一般的情况 (9), 由于对任一方阵, 可取 y_0 充分大, 使 $(x_0-y_0)^2 - x_2^2 - x_3^2 > 0$, 因此有 y_1 使

$$(x_0-y_0)^2 = (x_1-y_1)^2 + x_2^2 + x_3^2.$$

这样的 y_0, y_1 不能使

$$(-x_0-y_0)^2 = (x_1-y_1)^2 + x_2^2 + x_3^2$$

成立. 因此 (9) 中的 $\pm$ 号只能取 $+$ 号, 因此每一元素都不变.

到此为止, 我们多次用了仿射变换, 最后化 Φ 为恒等变换, 也就是原来的 Φ 是仿射变换.

5.11　时空几何的基本定理

定理 1　一个把时空点变为时空点的一一对应, 而且使行列式

$$|X_1 - X_2| \tag{1}$$

不变, 一定是 Poincaré 变换, 即

$$Y = \pm AX\overline{A'} + B, \quad |A| = 1, \quad \overline{B'} = B, \quad Y = X'.$$

证　即使 (1) 不变, 当然就是粘切关系不变. 因此

$$Y = \rho AX\overline{A'} + B,$$

或

$$Y = \rho AX'\overline{A'} + B.$$

但

$$|Y_1 - Y_2| = \rho^2|X_1 - X_2| \cdot |A|^2 = \rho^2|X_1 - X_2|.$$

因此, 得 $\rho = \pm 1$.

定理 2　在定理 1 的假定下, 再加以时光不能倒流, 左旋、右旋的旋向不变, 则一定是如下的形式:

$$Y = AX\overline{A'} + B.$$

显然还可有: 因果关系不变是光速不变的推论.

5.12 H 方阵的射影几何学

在上面的处理中, 我们实际上有一个无言的约定:"把一个有限的 H 方阵变为一个有限的 H 方阵". 如果我们允许有无穷远的 H 方阵, 则我们有以下的"H 方阵射影几何学".

首先, 对一个 H 方阵, 我们引进"齐性坐标", 也就是把 X 表成为

$$X = X_2^{-1} X_1, \tag{1}$$

这里 X_1, X_2 都是二行二列的方阵. 这一对方阵

$$(X_1, X_2) \tag{2}$$

称为 X 的齐性坐标. 由于

$$X = \overline{X'},$$

所以

$$X_2^{-1} X_1 = \overline{X'}_1 \overline{X'}_2^{-1}, \qquad 即\ X_1 \overline{X}_2' = X_2 \overline{X}_1'. \tag{3}$$

或可写成为

$$(X_1, X_2) J \overline{(X_1, X_2)'} = 0, \qquad 而\ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{4}$$

以上处理中, 我们当然假定了 X_2 是非奇异的. 同时, 对所有的 $Q, |Q| \neq 0$,

$$Q(X_1, X_2) = (QX_1, QX_2) \tag{5}$$

代表同一个 X. 我们现在扩充这一概念, 如果二行四列方阵 (2) 的秩等于 2, 而且适合于 (4), 则 (X_1, X_2) 称为一 Hermite 对, 或简称 H 对. 如果两个 H 对相差一因子 (如 (5)), 则称为等价. 由等价关系把 H 对分类, 一类定义一点, 所有这样定义出来的点称为形成一 H 方阵的射影空间.

命 T 是一适合于

$$TJ\overline{T'} = J \tag{6}$$

的 4×4 方阵, 则

$$(X_1^*, X_2^*) = Q(X_1, X_2)T \tag{7}$$

称为射影空间的一个变换, 命

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{8}$$

则

$$A\overline{B'} = B\overline{A'}, \qquad C\overline{D'} = D\overline{C'}, \qquad A\overline{D'} - B\overline{C'} = I. \tag{9}$$

表成非齐次形式, 则变换 (7) 可以写成为

$$\begin{aligned} X^* &= X_2^{*-1}X_1^* = (X_1B + X_2D)^{-1}(X_1A + X_2C) \\ &= (XB+D)^{-1}(XA+C) = (\overline{A'}X + \overline{C'})(\overline{B'}X + \overline{D'})^{-1}. \end{aligned} \tag{10}$$

这可由 (9) 直接验证. 而 $|X_2| = 0$ 的点称为无穷远处的点. 由 (10) 可知

$$\begin{aligned} X^* - Y^* =&(XB+D)^{-1}(XA+C) \\ &- (\overline{A'}Y + \overline{C'})(\overline{B'}Y + \overline{D'})^{-1} \\ =&(XB+D)^{-1}[(XA+C)(\overline{B'}Y + \overline{D'}) \\ &- (XB+D)(\overline{A'}Y + \overline{C'})](\overline{B'}Y + \overline{D'})^{-1} \\ =&(XB+D)^{-1}(X-Y)(\overline{B'}Y + \overline{D'})^{-1}. \end{aligned} \tag{11}$$

因此, 粘切关系还是不变的 (如果 X_1^*, X_2^*, X_1, X_2 都是有限点).

对齐性坐标, 粘切条件可以写成为

$$|(X_1, X_2)J\overline{(Y_1, Y_2)}'| = 0. \tag{12}$$

我们可以证明

定理 1 把 H 方阵的射影空间一一对应地变为自己, 而且使粘切关系 (12) 不变的变换一定是 (7) 的形式所成的群, T 适合 (6), 再添上适合于

$$TJ\overline{T'} = -J$$

的 T, 还有变形

$$(X_1^*, X_2^*) = (\overline{X}_1, \overline{X}_2).$$

这定理的证明从略, 读者在了解了仿射几何的基本定理后, 不难自己补出.

Фок 在较强的假定下, 不必要地运用偏微分方程 (波前方程), 用了较高深的数学工具, 获得同样的结论. 他称之为 Möbius 变换, 即除去 H 方阵的仿射变换外, 还有

$$\begin{aligned} x_i^* =&[x_i - \alpha_i(x_0^2 - x_1^2 - x_2^2 - x_3^2)]/[1 - 2(\alpha_0x_0 - \alpha_1x_1 - \alpha_2x_2 - \alpha_3x_3) \\ &+(\alpha_0^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2)(x_0^2 - x_1^2 - x_2^2 - x_3^2)] \end{aligned}$$

运用

$$\begin{pmatrix} \alpha_0 + \alpha_1 & \alpha_2 + i\alpha_3 \\ \alpha_2 - i\alpha_3 & \alpha_0 - \alpha_1 \end{pmatrix}^{-1} = \frac{1}{\alpha_0^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2} \begin{pmatrix} \alpha_0 - \alpha_1 & -\alpha_2 - i\alpha_3 \\ -\alpha_2 + i\alpha_3 & \alpha_0 + \alpha_1 \end{pmatrix}$$

不难推出 Möbius 变换可以写成 (10) 的形式.

5.13 射影变换与因果关系

值得注意的是以下的特例: 射影变换 (Möbius 变换)

$$X^* = X\left(\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} X + I\right)^{-1}$$

这变换把 0 变为 0, 把 I 变为 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, 即把因果关系的原则破坏了. 更深刻些有以下的结果:

任何一个带有分母的变换 (10), 一定破坏因果关系, 先证明一个引理.

引理 命 H 是一给定的 H 方阵. 如果对任一适合于 $|X|>0$ 的 H 方阵, 常有 $|HX+I|>0$, 则 $H=0$.

证 不失去普遍性, 可取 $H=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $(\alpha \geqslant \beta, \alpha>0)$. 我们说如果 $H \neq 0$, 则可以找到一个 $X=\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \xi\eta>0$, 使

$$|XH+I|=(\alpha\xi+1)(\beta\eta+1)<0.$$

若 (1) $\beta<0$, 则取 ξ 为充分小正数, $\eta>\dfrac{1}{|\beta|}$ 就行了.

(2) $\beta=0$, 则取 $\xi<-1/\alpha$, $\eta<0$ 即得.

(3) 如果 $\alpha \geqslant \beta>0$, 则可取 $\varepsilon>0$,

$$\xi=-\frac{1+\varepsilon}{\alpha}, \qquad \eta=-\frac{1-\varepsilon}{\beta},$$

使

$$\xi\eta=\frac{1-\varepsilon^2}{\alpha\beta}>0,$$

而

$$(\alpha\xi+1)(\beta\eta+1)=[-(1+\varepsilon)+1][-(1-\varepsilon)+1]=-\varepsilon^2<0.$$

现在我们证明

定理 1 非仿射变换的射影变换一定破坏因果关系.

证 假定这一变换把 X, Y 变为 X^*, Y^*, 则

$$X^*-Y^*=(XB+D)^{-1}(X-Y)(\overline{B'}Y+\overline{D'})^{-1}.$$

仿射变换可以把任一点变为 0. 我们不妨假定 $Y = Y^* = 0$. 因此 $C = 0$. 因而 $|D| \neq 0$. 上式变为

$$X^* = (XB + D)^{-1} X \overline{D'}^{-1}.$$

于是

$$|X^*| = |XB + D|^{-1} |\overline{D'}|^{-1} |X| = |XBD^{-1} + I|^{-1} |D\overline{D'}|^{-1} |X|.$$

由于 5.12 节 (9) 可知 BD^{-1} 是一 H 方阵, 由引理即推出定理.

5.14 附　记

1. 共形变换的度量. 在 5.12 节 (11) 中命 $Y \to X$, 则得微分方阵逆变式

$$dX^* = (XB + D)^{-1} dX (\overline{B'} X + \overline{D'})^{-1},$$

取行列式, 得

$$dx_0^{*2} - dx_1^{*2} - dx_2^{*2} - dx_3^{*2} = \rho(x_0, x_1, x_2, x_3)^{-2} (dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2),$$

这里 $\rho = |XB + D|$. 这是“共形”名称的来源.

2. 如果用假定 (A), 但把假定 (B) 减弱为

(B′) 通过任一观察点的光速不变, 也就是光速不变的局部性. 也可以得出同样的结论来.

命仿射空间的速度矢量为 (v_1, v_2, v_3). 其变换规律是

$$x_i^* = \sum_{j=0}^{3} a_{ij} x_j, \qquad i = 0, 1, 2, 3.$$

于是

$$v_i^* = \frac{dx_i^*}{dx_0^*} = \frac{a_{i0} + \sum_{j=1}^{3} a_{ij} v_j}{a_{00} + \sum_{j=1}^{3} a_{0j} v_j}, \qquad i = 1, 2, 3,$$

即成一三维的射影空间. 任一观察点, 观察到光速不变也就是使

$$v_1^2 + v_2^2 + v_3^2 = 1 \tag{1}$$

不变 (光速为 1), 使 (1) 不变也就是对所有的适合于 (1) 的 v_1, v_2, v_3 常有

$$\left(a_{00} + \sum_{j=1}^{3} a_{0j} v_j\right)^2 = \sum_{i=1}^{3} \left(a_{i0} + \sum_{j=1}^{3} a_{ij} v_j\right)^2. \tag{2}$$

取 $v_1 = \pm 1,\ v_2 = v_3 = 0$, 则得

$$(a_{00} \pm a_{01})^2 = \sum_{i=1}^{3}(a_{i0} \pm a_{i1})^2.$$

取 $\pm$ 号相加减得

$$a_{00}a_{01} = \sum_{i=1}^{3} a_{i0}a_{i1},$$

$$a_{00}^2 + a_{01}^2 = \sum_{i=1}^{3}(a_{i0}^2 + a_{i1}^2).$$

同法可得

$$a_{00}a_{0j} = \sum_{i=1}^{3} a_{i0}a_{ij}, \qquad j = 2, 3.$$

$$a_{00}^2 + a_{0j}^2 = \sum_{i=1}^{3}(a_{i0}^2 + a_{ij}^2), \qquad j = 2, 3.$$

因之, 得到

$$\begin{cases} -a_{0j}^2 + \sum\limits_{i=1}^{3} a_{ij}^2 = a_{00}^2 - \sum\limits_{i=1}^{3} a_{i0}^2 = \rho, & j = 1, 2, 3. \\ a_{00}a_{0j} - \sum\limits_{i=1}^{3} a_{i0}a_{ij} = 0, & j = 1, 2, 3. \end{cases}$$

即得

$$X^* = XL,$$

而 $L = L^{(4,4)}$ 适合于

$$L[1, -1, -1, -1]L' = \rho[1, -1, -1, -1].$$

3. 所有小于光速的矢量 (v_1, v_2, v_3) 成一空间

$$v_1^2 + v_2^2 + v_3^2 < 1.$$

它的变换群就是 2 中所提到的群. 在第 7 讲中将讲二维的相仿的空间.

第 6 讲　非欧几何学

6.1　扩充空间的几何性质

前已说明我们所讨论的群 G 是由以下的变换所形成的:

$$y = \frac{xT + xx'v_1 + v_2}{xu_2' + xx'b + d} \tag{1}$$

(同时有

$$yy' = \frac{xu_1' + xx'a + c}{xu_2' + xx'b + d}\Bigg). \tag{2}$$

记

$$M = \begin{pmatrix} T & u_1' & u_2' \\ v_1 & a & b \\ v_2 & c & d \end{pmatrix} \tag{3}$$

适合于

$$MJM' = J. \tag{4}$$

则齐次坐标就是

$$(\xi^*, \eta_1^*, \eta_2^*) = \rho(\xi, \eta_1, \eta_2)M, \tag{5}$$

这里 M 是使

$$\xi\xi' - \eta_1\eta_2 = 0$$

不变的变形, 命 $\eta_1 = s_1 + s_2$, $\eta_2 = -s_1 + s_2$, 则得

$$\xi\xi' + s_1^2 - s_2^2 = 0.$$

以 s_2 除之, 则得一个 $n+1$ 维的单位球. 因此, 我们所研究的经过 G 群而扩充的 n 维的空间的研究与使 $n+1$ 维空间的单位球不变的球面几何学是等价的, 这种几何我们将来研究混合型偏微分方程再谈.

这就是把复平面与球面作对应的球面投影法的推广.

我们也已知在变形 G 下球分三类: 实、点、虚. 而且它们可以各变为以下的标准型:

(i) $xx'=1$ (实球);

(ii) $xx'=0$ (点球);

(iii) $xx'=-1$ (虚球).

固定一个球称为绝对, 使这球不变的诸变换成一群以 H 表之, H 群下的几何学称为非欧几何学, 对应于 (i), (ii), (iii) 分为三类几何学: 双曲、抛物与椭圆.

为了引用 (4) 式方便起见, 我们把它所包含的关系具体写出来:

$$TT'-\frac{1}{2}(u_1'u_2+u_2'u_1)=I^{(n)}, \tag{6}$$

$$Tv_1'-\frac{1}{2}(u_1'b+u_2'a)=0, \tag{7}$$

$$Tv_2'-\frac{1}{2}(u_1'd+u_2'c)=0, \tag{8}$$

$$v_1v_1'-ab=0, \tag{9}$$

$$v_1v_2'-\frac{1}{2}(ad+bc)=-\frac{1}{2}, \tag{10}$$

$$v_2v_2'-cd=0. \tag{11}$$

由 (4) 取逆, 得到

$$M'J^{-1}M=J^{-1},\quad J^{-1}=\begin{pmatrix} I & 0 & 0\\ 0 & 0 & -2\\ 0 & -2 & 0\end{pmatrix}.$$

即得

$$T'T-2(v_1'v_2+v_2'v_1)=I, \tag{12}$$

$$T'u_1'-2(v_1'c+v_2'a)=0, \tag{13}$$

$$T'u_2'-2(v_1'd+v_2'b)=0, \tag{14}$$

$$u_1u_1'-4ac=0, \tag{15}$$

$$u_1u_2'-2(ad+bc)=-2, \tag{16}$$

$$u_2u_2'-4bd=0. \tag{17}$$

6.2 抛物几何学

这是使一个点球不变的群下所定义出来的几何学, 不妨假定这点是 $x=\infty$, 由 (1.1) 可知 $b=0$, 再由 (1.9) 与 (1.17) 可知

$$v_1=0,\quad u_2=0. \tag{1}$$

再由 (1.6) 得

$$TT' = I. \tag{2}$$

由 (1.16), $ad = 1$, 即

$$a = \frac{1}{d}.$$

再由 (1.8) 可知

$$u_1 = \frac{2}{d}v_2T' \tag{3}$$

最后由 (1.15) 得

$$c = \frac{4}{d}v_2v_2'. \tag{4}$$

(1), (2), (3), (4) 完全决定了变形的形式, 即

$$M = \begin{pmatrix} T & \frac{2}{d}Tv_2' & 0 \\ 0 & \frac{1}{d} & 0 \\ v_2 & \frac{4}{d}v_2v_2' & d \end{pmatrix}.$$

即得抛物几何学中变形的一般形式

$$y = a(xT + v). \tag{5}$$

这是由旋转 (及反演)

$$y = xT.$$

平移

$$y = x + v,$$

及放大 (及缩小)

$$y = ax$$

所组成的.

抛物几何学的确定定义是:

空间: 所有的有限点.

群: 由旋转、反演、平移与放大所演出的群.

如果除去放大不论, 这几何就是 n 维的欧几里得几何.

我们不深入讨论这种几何学.

6.3　椭圆几何学

虚球可以用矢量

$$(0;1,1)$$

表之. 我们的群是由适合于

$$(0;1,1)=\rho(0;1,1)M'$$

的方阵 M 所组成的.

换变数

$$\eta_1=s_1+s_2,\quad \eta_2=-s_1+s_2,$$

我们把 M 换成为 N 它使

$$N[1,1,\cdots,1,-1]N'=[1,\cdots,1,-1], \tag{1}$$

而矢量 (0;1,1) 变为 ($s_1=0,\ s_2=1$)

$$(0;0,1),$$

即

$$(0,0,\cdots,0,1)N=\rho(0,0,\cdots,0,1).$$

因此

$$N=\begin{pmatrix} N_1^{(n+1)} & w_1' \\ 0 & d \end{pmatrix},$$

由 (1) 得 $w_1=0,\ d=\pm 1$, 及

$$N_1N_1'=I^{(n+1)},$$

这 N_1 就是 $n+1$ 行列的正交方阵.

因此, 椭圆几何就是 $n+1$ 维球面上的几何学, 它是在旋转、反演所造成的群 T 的几何学.

6.4　双曲几何学

矢量

$$(0,\cdots,0;1,-1)$$

代表单位球, 我们的群是由适合于

$$(0,\cdots,0,1,-1)=\rho(0,0,\cdots,0,1,-1)M'$$

的方阵 M 所组成的, 也就是由适合于

$$u_1=u_2 \tag{1}$$

及

$$a-b+c-d=0 \tag{2}$$

的变形所组成的, 这也可以从 (1.2) 中取 $xx'=yy'=1$ 而推出.

命 $u_1=u_2=u$, 由 (1.6) 可知

$$TT'=I^{(n)}+u'u. \tag{3}$$

即 T 非奇异的, 再由 (1.7), (1.8) 得出

$$\begin{cases} v_1=\dfrac{1}{2}(a+b)uT'^{-1}, \\ v_2=\dfrac{1}{2}(c+d)uT'^{-1}. \end{cases} \tag{4}$$

再由 (1.9) 及 (3) 可知

$$\frac{1}{4}(a+b)^2uT'^{-1}T^{-1}u=ab,$$

即

$$(a+b)^2u(I+u'u)^{-1}u'=4ab,$$
$$(a+b)^2uu'(1+uu')^{-1}=4ab.$$

即得

$$(a-b)^2uu'=4ab. \tag{5}$$

同法从 (1.10) 与 (1.11) 得出

$$[(a-b)(c-d)+2]uu'=-2+2(ad+bc), \tag{6}$$
$$(c-d)^2uu'=4cd. \tag{7}$$

从 (2), (5), (6), (7) 解得

$$\begin{cases} a=d=\dfrac{1}{2}(\pm1\pm\sqrt{1+uu'}), \\ b=c=\dfrac{1}{2}(\mp1\pm\sqrt{1+uu'}). \end{cases} \tag{8}$$

$u=0$ 的情况是极易处理的, 我们假定 $u\neq 0$, 由于 $\pm M$ 代表同一变换, 不妨假定 $a>0$, 即

$$a=d=\frac{1}{2}(\pm 1+\sqrt{1+uu'}),$$
$$b=c=\frac{1}{2}(\mp 1+\sqrt{1+uu'}).$$

再由

$$1-yy'=\frac{(a-b)(1-xx')}{xu'+xx'b+a},$$

可知把单位球内部变为内部的变化是

$$a=d=\frac{1}{2}(1+\sqrt{1+uu'}),$$
$$b=c=\frac{1}{2}(-1+\sqrt{1+uu'}), \tag{9}$$

也就是 M 中的元素必须适合 (3), 而 v_1, v_2, a, b, c, d 由 (4) 及 (9) 表出之.

这也不但算出了这一群的参变量 u 有 n 个, u 固定了, T 有 $\frac{1}{2}n(n-1)$ 个, 即一共有 $\frac{1}{2}n(n+1)$ 个参数 $\left(\text{正交方阵的参数等于 } \frac{1}{2}n(n-1)\right)$.

总之, 群 G 内使单位球内部变为内部的变形可以写成为

$$y=\frac{xT+\frac{1}{2}(1+xx')\sqrt{1+uu'}uT'^{-1}}{xu'+\frac{1}{2}(1+xx')\sqrt{1+uu'}+\frac{1}{2}(1-xx')}. \tag{10}$$

由于

$$uT'^{-1}=u(I+u'u)^{-1}T=(1+uu')^{-1}uT,$$

故可以改写为

$$y=\frac{x+\frac{1}{2}(1+xx')(1+uu')^{-\frac{1}{2}}u}{xu'+\frac{1}{2}(1+xx')(1+uu')^{\frac{1}{2}}+\frac{1}{2}(1-xx')}T.$$

使 0 点不变的变形是 $u=0$ 的变形, 即

$$y=xT,\quad TT'=I, \tag{11}$$

因此, 一般的变形确是由第 1 讲中所讲到的变形所演出的. 因为任何变换可能把 u 变为 0, 由第 1 讲中的变换能把 a 变为 0, 因此, 只需考虑把 0 变为 0 的变换的一般形式即足, 而它确是 (10). 也是第 1 讲中所提起过的.

6.5 测 地 线

定理 1 过任两点有一而且唯一的一条测地线. 具体地讲, 命 x_0, x_0^* 是两点, 则沿测地线的积分

$$\int_{x_0}^{x_0^*} \frac{\sqrt{dx dx'}}{1 - xx'}$$

最短, 其他的都大于它.

证 不失普遍性, 可取 $x_0 = 0$ 及 $x_0^* = (\delta, 0, \cdots, 0)$, 因为由可递性不妨假定 $x_0 = 0$, 又行一旋转可以假定 $x_0^* = (\delta, 0, \cdots, 0)$.

命

$$x = x(t), \quad 0 \leqslant t \leqslant 1$$

是联这两点的曲线, 即

$$x(0) = 0, \quad x(1) = (\delta, 0, \cdots, 0),$$

则积分等于

$$\begin{aligned}\int_0^1 \frac{\sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}}{1 - x_1^2 \cdots - x_n^2} dt &\geqslant \int_0^1 \frac{\frac{dx_1}{dt}}{1 - x_1^2} dt \\ &= \frac{1}{2} \log \frac{1 + x_1(t)}{1 - x_1(t)} \Bigg|_0^1 = \frac{1}{2} \log \frac{1+\delta}{1-\delta}.\end{aligned}$$

即得所证.

第 7 讲　混合型偏微分方程

从二维出发, 但熟悉线性代数的读者可以直接推到高维.

7.1　实射影平面

还是从单位圆开始, 先研究使单位圆不变的射影变换.

我们所讨论的群是由使

$$x^2 + y^2 < 1 \tag{1}$$

变为其自己的射影变换

$$x_1 = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad y_1 = \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3} \tag{2}$$

所组成的, 也就是方阵

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

是满秩的, 而且是适合于

$$A' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} A = \rho \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3}$$

的, 由于 (2) 的齐次性, 不妨假定 $\rho = \pm 1$, 再取 (3) 的行列式, 易见 $\rho = 1$, 今后我们常假定 $\rho = 1$.

这个群记作 Γ, 是由以下的一些元素所演成的:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos h\psi & \sin h\psi \\ 0 & \sin h\psi & \cos h\psi \end{pmatrix}$$

称为旋转、反射与双曲旋转. 或者更具体些, 它们可以写成为

$$\begin{cases} x_1 = x\cos\theta + y\sin\theta, \\ y_1 = -x\sin\theta + y\cos\theta, \end{cases} \tag{4}$$

$$\begin{cases} x_1 = x, \\ y_1 = -y. \end{cases} \tag{5}$$

及对实数 $\mu(-1 < \mu < 1)$,

$$\begin{cases} x_1 = \sqrt{1-\mu^2}x/(1-\mu y), \\ y_1 = (y-\mu)/(1-\mu y), \end{cases} \tag{6}$$

或

$$\begin{cases} x_1 = x/(y\sin h\psi + \cos h\psi), \\ y_1 = (y\cos h\psi + \sin h\psi)/(y\sin h\psi + \cos h\psi). \end{cases}$$

现在来证明这一点, 在 A 的左右各乘以一个类型 (4) 的方阵, 可以使 $b_1 = a_2 = 0$. 如此便不难推出 A 成为

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{pmatrix},$$

这显然是 (5) 与 (6) 的乘积.

在群 Γ 之下, 圆内一点可以变为其中的任一点, 要证明这点并不困难, 首先经过旋转任一点可以变为 $(0,\lambda)(\lambda > 0)$. 当 $\lambda < 1$, 可以由 (6) 变为 (0,0)(实质上, 圆外之点也成一可递集合).

我们现在研究群 Γ 下的微分不变量, 联两点 (x,y), $(x+dx,\ y+dy)$ 的直线是

$$(x+\lambda dx,\ y+\lambda dy),$$

这直线与单位圆的交点, 可由

$$(x+\lambda dx)^2 + (y+\lambda dy)^2 = 1$$

来决定, 即

$$\lambda^2(dx^2+dy^2) + 2\lambda(xdx+ydy) + x^2+y^2-1 = 0.$$

这式子的判别式是

$$(xdx+ydy)^2 - (dx^2+dy^2)(x^2+y^2-1),$$

即

$$(1-y^2)dx^2 + 2xydxdy + (1-x^2)dy^2.$$

这建议了这微分二次型可能是一个共变量, 实际计算, 这的确是一个共变量, 而且

$$\frac{(1-y^2)dx^2+2xydxdy+(1-x^2)dy^2}{(1-x^2-y^2)^2} \tag{A}$$

是一个不变量.

这一性质当然可以从交比推出 (即由二点及与圆的二交点的交比推出), 但也可以直接证之如下:

(A) 的分子等于

$$dx^2+dy^2-(ydx-xdy)^2,$$

这显然是经旋转与反射而不变的, 现在进一步证明, 它经 (6) 而共变,

$$dx_1=\frac{\sqrt{1-\mu^2}}{1-\mu y}dx+\frac{\mu\sqrt{1-\mu^2}}{(1-\mu y)^2}xdy,$$
$$dy_1=\frac{1-\mu^2}{(1-\mu y)^2}dy.$$

显然可见

$$\begin{aligned}&dx_1^2+dy_1^2-(x_1dy_1-y_1dx_1)^2\\=&\frac{1}{(1-\mu y)^4}[(1-\mu^2)\{(1-\mu y)dx+\mu xdy\}^2+(1-\mu^2)^2dy^2]\\&-\frac{1-\mu^2}{(1-\mu y)^6}[x(1-\mu^2)dy-(y-\mu)\{(1-\mu y)dx+\mu xdy\}]^2\\=&\frac{1-\mu^2}{(1-\mu y)^4}[\{(1-\mu y)dx+\mu xdy\}^2+(1-\mu^2)dy^2-\{xdy-(y-\mu)dx\}^2]\\=&\frac{(1-\mu^2)^2}{(1-\mu y)^4}[(1-y^2)dx^2+2xydxdy+(1-x^2)dy^2],\end{aligned} \tag{7}$$

而另一方面,

$$\begin{aligned}1-x_1^2-y_1^2&=1-\frac{1}{(1-\mu y)^2}[(1-\mu^2)x^2+(y-\mu)^2]\\&=\frac{1-\mu^2}{(1-\mu y)^2}(1-x^2-y^2).\end{aligned} \tag{8}$$

所以得到

$$\begin{aligned}&\frac{(1-y_1^2)dx_1^2+2x_1y_1dx_1dy_1+(1-x_1^2)dx_1^2}{(1-x_1^2-y_1^2)^2}\\=&\frac{(1-y^2)dx^2+2xydxdy+(1-x^2)dx^2}{(1-x^2-y^2)^2}.\end{aligned} \tag{9}$$

这微分不变式作为我们的 Riemann 度量.

7.2 偏微分方程

与 (A)“对偶”的有以下的二阶偏微分算子

$$\begin{aligned}\Delta u =&(1-x^2-y^2)\left[(1-x^2)\frac{\partial^2}{\partial x^2}-2xy\frac{\partial^2}{\partial x\partial y}\right.\\&\left.+(1-y^2)\frac{\partial^2}{\partial y^2}-2x\frac{\partial}{\partial x}-2y\frac{\partial}{\partial y}\right]u.\end{aligned}\tag{B}$$

这也是经 Γ 而不变的.

要证明这一点有两种方法: 一种是把 (A) 看成为 Riemann 度量, 这一 Riemann 空间的 Lamé 算子 (或 Bertrami 算子) 就是 (B), 因而由一般性的定理, 得出这一性质. 计算也是冗长的, 反而不如直接代入的简捷, 我们现在用直接代入法, 易证这算子是经 7.1 节的变换 (4), (5) 而不变的, 现在证明, 它也经 7.1 节的 (6) 而不变, 现在有

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial x_1}\frac{\sqrt{1-\mu^2}}{1-\mu y},\\\frac{\partial u}{\partial y}&=\frac{\partial u}{\partial x_1}\frac{\mu x\sqrt{1-\mu^2}}{(1-\mu y)^2}+\frac{\partial u}{\partial y_1}\frac{1-\mu^2}{(1-\mu y)^2}.\end{aligned}$$

因此得出

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial^2 u}{\partial x_1^2}\frac{1-\mu^2}{(1-\mu y)^2},\\\frac{\partial^2 u}{\partial x\partial y}&=\frac{\partial^2 u}{\partial x_1^2}\frac{\mu(1-\mu^2)x}{(1-\mu y)^3}+\frac{\partial^2 u}{\partial x_1\partial y_1}\frac{(1-\mu^2)^{3/2}}{(1-\mu y)^3}+\frac{\partial u}{\partial x_1}\frac{\mu\sqrt{1-\mu^2}}{(1-\mu y)^2},\end{aligned}$$

及

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}=&\frac{\partial^2 u}{\partial x_1^2}\frac{\mu^2(1-\mu^2)x^2}{(1-\mu y)^4}+2\frac{\partial^2 u}{\partial x_1\partial y_1}\frac{\mu x(1-\mu^2)^{3/2}}{(1-\mu y)^4}\\&+\frac{\partial^2 u}{\partial y_1^2}\frac{(1-\mu^2)^2}{(1-\mu y)^4}+2\frac{\partial u}{\partial x_1}\frac{\mu^2 x\sqrt{1-\mu^2}}{(1-\mu y)^3}+2\frac{\partial u}{\partial y_1}\frac{\mu(1-\mu^2)}{(1-\mu y)^3},\end{aligned}$$

因此

$$\begin{aligned}&(1-x^2)\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+(1-y^2)\frac{\partial^2 u}{\partial y^2}-2x\frac{\partial u}{\partial x}-2y\frac{\partial u}{\partial y}\\=&\frac{\partial^2 u}{\partial x_1^2}\left[(1-x^2)\frac{1-\mu^2}{(1-\mu y)^2}-\frac{2\mu(1-\mu^2)x^2y}{(1-\mu y)^3}+(1-y^2)\frac{\mu^2x^2(1-\mu^2)}{(1-\mu y)^4}\right]\end{aligned}$$

$$
\begin{aligned}
&+2\frac{\partial^2 u}{\partial x_1\partial y_1}\left[-\frac{(1-\mu^2)^{3/2}xy}{(1-\mu y)^3}+\frac{\mu(1-\mu^2)^{3/2}}{(1-\mu y)^4}(1-y^2)\right]\\
&+\frac{\partial^2 u}{\partial y_1^2}(1-y)^2\frac{(1-\mu^2)^2}{(1-\mu y)^4}-2\frac{\partial u}{\partial x_1}\left[\frac{xy\mu\sqrt{1-\mu^2}}{(1-\mu y)^2}\right.\\
&\left.-\frac{\mu^2x^2\sqrt{1-\mu^2}}{(1-\mu y)^3}(1-y^2)+x\frac{\sqrt{1-\mu^2}}{1-\mu y}+y\frac{\mu x\sqrt{1-\mu^2}}{(1-\mu y)^2}\right]\\
&-2\frac{\partial u}{\partial y_1}\left[-\frac{(1-\mu^2)\mu}{(1-\mu y)^3}(1-y^2)+\frac{(1-\mu^2)y}{(1-\mu y)^2}\right]\\
=&\frac{1-\mu^2}{(1-\mu y)^2}\left\{\frac{\partial^2 u}{\partial x_1^2}\left[1-x^2-\frac{2\mu yx^2}{1-\mu y}+(1-y^2)\frac{\mu^2x^2}{(1-\mu y)^2}\right]\right.\\
&-2\frac{\partial^2 u}{\partial x_1\partial y_1}\left[\frac{(1-\mu^2)^{1/2}x}{1-\mu y}\left(y-\frac{\mu(1-y^2)}{1-\mu y}\right)\right]+\frac{\partial^2 u}{\partial y_1^2}\frac{(1-y^2)(1-\mu^2)}{(1-\mu y)^2}\\
&-2\frac{\partial u}{\partial x_1}(1-\mu^2)^{-1/2}x\left[(1-\mu y)+2\mu y-\frac{\mu^2(1-y^2)}{1-\mu y}\right]\\
&\left.-2\frac{\partial u}{\partial y_1}\left[y-\frac{\mu}{1-\mu y}(1-y^2)\right]\right\}\\
=&\frac{1-\mu^2}{(1-\mu y)^2}\left\{\frac{\partial^2 u}{\partial x_1^2}\left(1-\frac{(1-\mu^2)x^2}{(1-\mu y)^2}\right)\right.\\
&-2\frac{\partial^2 u}{\partial x_1\partial y_1}\frac{(1-\mu^2)^{1/2}x(y-\mu)}{(1-\mu y)^2}\\
&\left.+\frac{\partial^2 u}{\partial y_1^2}\left[1-\left(\frac{y-\mu}{1-\mu y}\right)^2\right]-2\frac{\partial u}{\partial x_1}\frac{(1-\mu^2)^{1/2}x}{1-\mu y}-2\frac{\partial u}{\partial y_1}\frac{y-\mu}{1-\mu y}\right\}\\
=&\frac{1-\mu^2}{(1-\mu y)^2}\left\{\left(1-x_1^2\right)\frac{\partial^2 u}{\partial x_1^2}-2x_1y_1\frac{\partial^2 u}{\partial x_1\partial y_1}\right.\\
&\left.+(1-y_1^2)\frac{\partial^2 u}{\partial y_1^2}-2x_1\frac{\partial u}{\partial x_1}-2y\frac{\partial u}{\partial y_1}\right\}.
\end{aligned}\tag{1}
$$

又由 7.1 节 (8), 可得 (B) 的不变性.

注意, 这变形的 Jacobian 是

$$
\frac{\partial(x_1,y_1)}{\partial(x,y)}=\left(\frac{\sqrt{1-\mu^2}}{1-\mu y}\right)^3.
$$

在单独考虑又带因子 $(1-x^2-y^2)$ 的 (A) 和 (B) 时, 我们必须注意, 所写出的共变因子是 Jacobian 的非整数乘方指数, 而偏微分方程

$$
(1-x^2)\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+(1-y^2)\frac{\partial^2 u}{\partial y^2}-2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=0 \tag{C}
$$

是以 (A) 为特征线的偏微分方程, 其特征线是单位圆的切线.

这个方程是混合型的, 在单位圆内是椭圆型, 在单位圆外是双曲型, 单位圆是变形线.

在射影变换之下, 单位圆是等价于任何非奇异实二次曲线的, 所以实际上我们处理了任何以非奇异二次曲线的变形线的一个问题.

这方程极坐标形式是

$$(1-\rho^2)\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2}+\left(\frac{1}{\rho}-2\rho\right)\frac{\partial u}{\partial \rho}=0, \tag{D}$$

也可以写成为

$$\rho\left|1-\rho^2\right|^{1/2}\frac{\partial}{\partial \rho}\left(\frac{\rho(1-\rho^2)}{|1-\rho^2|^{1/2}}\frac{\partial u}{\partial \rho}\right)+\frac{\partial^2 u}{\partial \theta^2}=0.$$

命 $\mathscr{D}$ 是射影平面上的一个域, 如果一个函数 $u=u(x,y)$ 在 $\mathscr{D}$ 上适合于偏微分方程 (C) 或 (D), 则 $u(x,y)$ 可以称为 $\mathscr{D}$ 上的调和函数. 关于 $u(x,y)$ 的一些应有的附加条件以后再说.

Lamé 算子的来源是求曲线坐标的位函数, 因此, 这样概念的引进是极自然的.

如果 $\mathscr{D}$ 在单位圆内, 即就得出通常的椭圆型方程. 如果 $\mathscr{D}$ 全在圆外, 那就是普通的双曲型偏微分方程. 我们现在集中注意于 $\mathscr{D}$ 的一部分在圆内另一部分在圆外的情况, 也就是讨论混合型偏微分方程的问题.

7.3 特 征 线

微分方程

$$(1-y^2)dx^2+2xydxdy+(1-x^2)dy^2=0 \tag{1}$$

的解称为特征线, 我们现在来解出这个微分方程式.

$x=1$ 显然是 (1) 的一个解, 这是单位圆的一条切线, 经过群 Γ 就得出单位圆的所有的切线. 因此, 单位圆的所有的切线都适合于微分方程 (1), 它们也就是 (1) 的通解, 而单位圆是这方程的奇解.

单位圆的切线是特征线, 而这些特征线的包络线也就是这单位圆.

特征线的一般形式是

$$x\cos\alpha+y\sin\alpha=1,$$

即得

$$y^2(1-\cos^2\alpha)=(1-x\cos\alpha)^2,$$
$$(x^2+y^2)\cos^2\alpha-2x\cos\alpha+1-y^2=0,$$

所以

$$\cos\alpha = \frac{x \pm \sqrt{x^2 - (x^2+y^2)(1-y^2)}}{x^2+y^2} = \frac{x \pm y\sqrt{x^2+y^2-1}}{x^2+y^2}.$$

因而推出方程 (C) 有如下形式的通解：

$$u(x,y) = f_1\left(\frac{x + y\sqrt{x^2+y^2-1}}{x^2+y^2}\right) + f_2\left(\frac{x - y\sqrt{x^2+y^2-1}}{x^2+y^2}\right),$$

此处 f_1 与 f_2 是两个任意函数, 在极坐标时, 通解形式是

$$g_1\left(\theta + \arccos\frac{1}{\rho}\right) + g_2\left(\theta - \arccos\frac{1}{\rho}\right).$$

如果

$$u(x,y)$$

是方程 (C) 的一个解, 则

$$u(x\cos\psi + y\sin\psi, -x\sin\psi + y\cos\psi)$$

也是一个解, 因而

$$u_1(x,y) = \frac{1}{2\pi}\int_0^{2\pi} u(x\cos\psi + y\sin\psi, -x\sin\psi + y\cos\psi)d\psi$$

也是一个解, 这样的解显然是经过旋转而不变的, 这一函数显然仅是 ρ 的函数, 而与 θ 无关. 先考虑这样的解, 由 (D) 得

$$\frac{\partial}{\partial\rho}\left(\frac{\rho(1-\rho^2)}{|1-\rho^2|^{1/2}}\frac{\partial u}{\partial\rho}\right) = 0, \quad \frac{\rho(1-\rho^2)}{|1-\rho^2|^{1/2}}\frac{\partial u}{\partial\rho} = C_1,$$

$$u = \begin{cases} C_1 \log\dfrac{1+\sqrt{1-\rho^2}}{\rho} + C_2, & \text{当}\rho \leqslant 1, \\ C_1 \arccos\dfrac{1}{\rho} + C_2, & \text{当}\rho > 1. \end{cases}$$

7.4　这偏微分方程与 Лаврентьев 方程的关系

在极坐标方程

$$\rho^2(1-\rho^2)\frac{\partial^2 u}{\partial\rho^2} + \rho(1-2\rho^2)\frac{\partial u}{\partial\rho} + \frac{\partial^2 u}{\partial\theta^2} = 0 \qquad \text{(D)}$$

中换变数 $\xi = f(\rho)$, 如此则得

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial \xi} f'(\rho), \quad \frac{\partial^2 u}{\partial \rho^2} = \frac{\partial^2 u}{\partial \xi^2} f'^2(\rho) + \frac{\partial u}{\partial \xi} f''(\rho),$$

代入 (D) 式得

$$\rho^2(1-\rho^2)f'^2(\rho)\frac{\partial^2 u}{\partial \xi^2} + [\rho^2(1-\rho^2)f''(\rho) + \rho(1-2\rho^2)f'(\rho)]\frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

我们取 $f(\rho)$ 使

$$\rho^2|1-\rho^2|(f'(\rho))^2 = 1. \tag{1}$$

微分此式得

$$2f'(\rho)f''(\rho)\rho^2|1-\rho^2| + \{2\rho|1-\rho^2| - 2\rho^2\text{sgn}(1-\rho^2)\rho\}f'^2(\rho) = 0,$$

即

$$\rho^2(1-\rho^2)f''(\rho) + \rho(1-2\rho^2)f'(\rho) = 0. \tag{2}$$

如果 $f(\rho)$ 适合于 (1), 则 (D) 式变为

$$\rho^2(1-\rho^2)f'^2(\rho)\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \theta^2} = 0,$$

即

$$\text{sgn}(1-\rho^2)\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \theta^2} = 0. \tag{3}$$

因此, 我们现在来解 (1) 式, 即取 $\xi = f(\rho)$ 使

$$\frac{d\xi}{d\rho} = -\frac{1}{\rho|1-\rho^2|^{1/2}},$$

解得

$$\xi = \begin{cases} \log \dfrac{1+\sqrt{1-\rho^2}}{\rho} + C, & \text{当 } \rho < 1, \\ -\arccos\dfrac{1}{\rho} + C', & \text{当 } \rho > 1. \end{cases}$$

取 $C = C' = 0$, 即得变形

$$\xi = \begin{cases} \operatorname{arc\,cosh} \dfrac{1}{\rho}, & \text{当 } 0 \leqslant \rho \leqslant 1, \\ -\arccos \dfrac{1}{\rho}, & \text{当 } \rho \geqslant 1. \end{cases} \tag{E}$$

这变形是连续的，而且变化的情况是

ρ	0		1		∞
ξ	∞	$\searrow$	0	$\searrow$	$-\frac{\pi}{2}$

经这样的变换后，所考虑的偏微分方程就变为

$$\operatorname{sgn}\xi\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\theta^2}=0,$$

此处 $-\frac{\pi}{2}\leqslant\xi\leqslant\infty$ 及 $-\pi<\theta\leqslant\pi$.

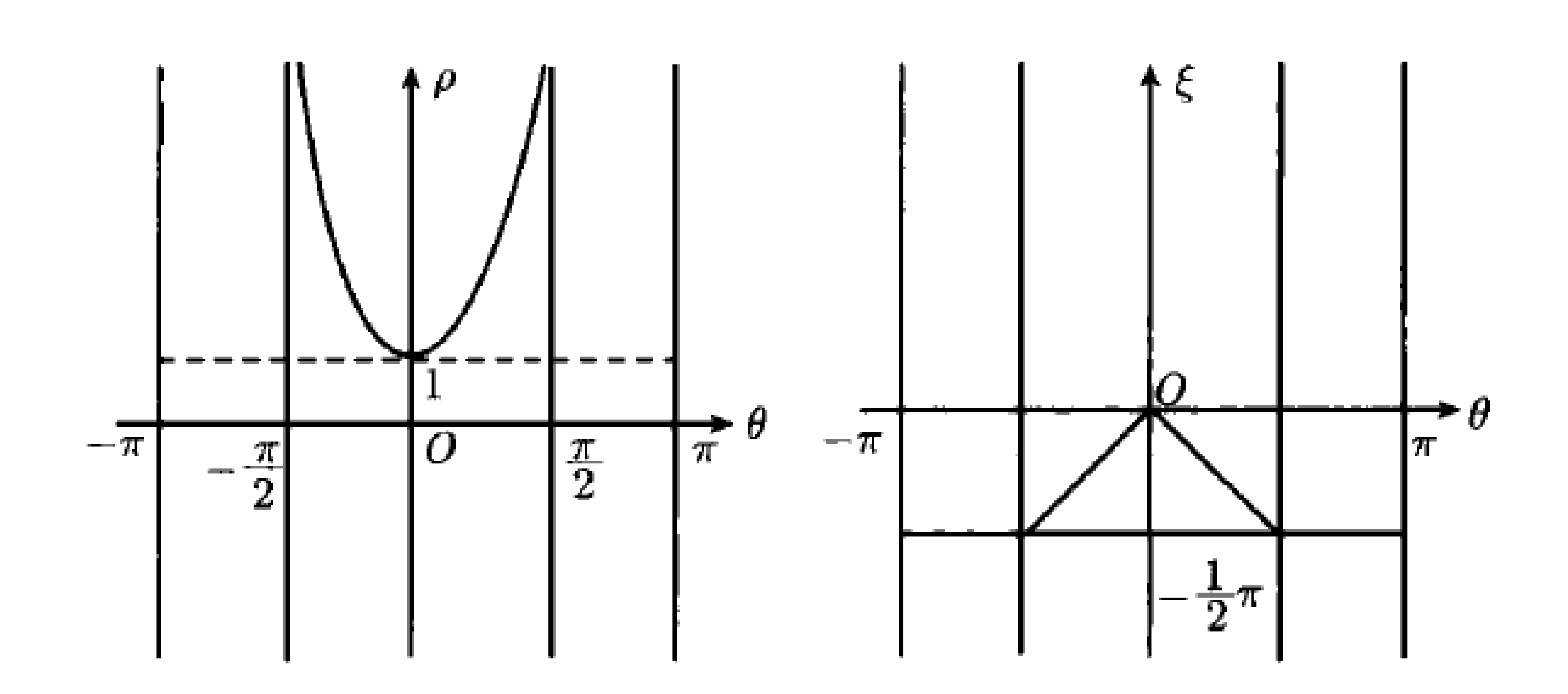
由 (E) 得

$$\rho=\begin{cases}\dfrac{1}{\cosh\xi}, & 当\ \xi\geqslant 0,\\ \dfrac{1}{\cos\xi}, & 当\ -\dfrac{\pi}{2}\leqslant\xi\leqslant 0.\end{cases} \tag{E$'$}$$

这一变换是连续的，而且有一级连续微商，但是它的二级微商在 $\xi=0$ 时不连续.

这也指明了在研究Лаврентьев方程的时候，在变形线上不能假定有二阶连续微商的道理.

如果把 (ρ,θ) 及 (ξ,θ) 都看成为垂直坐标，则得下图.



在 (ρ,θ) 平面上所考虑的区域是 $\rho>0$, $-\pi<\theta\leqslant\pi$ 的半条形，适合于 $0<\rho\leqslant 1$ 的部分是椭圆区，而适合于 $\rho>1$ 的部分是双曲区，所画的 U 形曲线是一条特征线. 其他的特征线可由此线水平移动得来. 但需注意，我们把二直线 $\theta=\pi$ 与 $\theta=-\pi$ 等同起来，在 (ξ,θ) 平面上，所考虑的区域点 $-\frac{1}{2}\pi<\xi$ 及 $-\pi<\theta\leqslant\pi$, 其中 $\xi<0$ 的部分是双曲区，所画出的 Λ 形曲线是特征线之一，其他可由水平移动得来.

7.5 分离变数法

现在研究方程

$$\rho^2(1-\rho^2)\frac{\partial^2 u}{\partial\rho^2}+\rho(1-2\rho^2)\frac{\partial u}{\partial\rho}=-\frac{\partial^2 u}{\partial\theta^2} \tag{D}$$

形如 $u=\varphi(\rho)\psi(\theta)$ 的解, 如此则

$$\frac{\rho^2(1-\rho^2)\varphi''(\rho)+\rho(1-2\rho^2)\varphi'(\rho)}{\varphi(\rho)}=-\frac{\psi''(\theta)}{\psi(\theta)}. \tag{1}$$

由于 θ 的周期性可知

$$-\frac{\psi''(\theta)}{\psi(\theta)}=n^2, \tag{2}$$

此处 n 是整数而得出 $\psi(\theta)=\cos n\theta$ 或 $\sin n\theta$.

由 (1) 及 (2) 得出

$$\rho^2(1-\rho^2)\varphi''+\rho(1-2\rho^2)\varphi'-n^2\varphi=0. \tag{3}$$

命 $\varphi=\rho^{-n}\Phi$, 得

$$\rho^2(1-\rho^2)(\rho^{-n}\Phi''-2n\rho^{-n-1}\Phi'+n(n+1)\rho^{-n-2}\Phi)$$
$$+\rho(1-2\rho^2)(\rho^{-n}\Phi'-n\rho^{-n-1}\Phi)-n^2\rho^{-n}\Phi=0,$$

即

$$\rho(1-\rho^2)\Phi''-(2n-1-2(n-1)\rho^2)\Phi'-n(n-1)\rho\Phi=0. \tag{4}$$

再命

$$\tau=1-\rho^2, \tag{5}$$

则得

$$\rho\tau\left(4\rho^2\frac{d^2\Phi}{d\tau^2}-2\frac{d\Phi}{d\tau}\right)+2\rho[1+2(n-1)\tau]\frac{d\Phi}{d\tau}-n(n-1)\rho\Phi=0,$$

即

$$4\tau(1-\tau)\frac{d^2\Phi}{d\tau^2}+2[1+2(n-1)\tau-\tau]\frac{d\Phi}{d\tau}-n(n-1)\Phi=0, \tag{6}$$

这是一个超几何微分方程, 它的解一般用

$$F\left(-\frac{1}{2}n,-\frac{1}{2}(n-1),\frac{1}{2},\tau\right)$$

表之, 但对这个特殊的方程, 我们可以直接验算这方程有以下的两个解

$$(1+\tau^{\frac{1}{2}})^n, \quad (1-\tau^{\frac{1}{2}})^n. \tag{7}$$

例如, 前者的第一第二阶微商各为

$$\frac{d}{d\tau}(1+\tau^{\frac{1}{2}})^n = \frac{1}{2}n(1+\tau^{\frac{1}{2}})^{n-1}\tau^{-\frac{1}{2}}$$

及

$$\frac{d^2}{d\tau^2}(1+\tau^{\frac{1}{2}})^n = \frac{1}{4}n(n-1)(1+\tau^{\frac{1}{2}})^{n-2}\tau^{-1} - \frac{1}{4}n(1+\tau^{\frac{1}{2}})^{n-1}\tau^{-\frac{3}{2}},$$

代入 (6) 式即得

$$\begin{aligned}
&(1-\tau)[n(n-1)(1+\tau^{\frac{1}{2}})^{n-2} - n(1+\tau^{\frac{1}{2}})^{n-1}\tau^{-\frac{1}{2}}] \\
&\quad + [1+2(n-1)\tau-\tau]n(1+\tau^{\frac{1}{2}})^{n-1}\tau^{-\frac{1}{2}} - n(n-1)(1+\tau^{\frac{1}{2}})^n \\
=&(1+\tau^{\frac{1}{2}})^{n-2}[n(n-1)(1-\tau) - n(1+\tau^{-\frac{1}{2}})(1-\tau) \\
&\quad + n(1+2(n-1)\tau-\tau)(1+\tau^{-\frac{1}{2}}) - n(n-1)(1+\tau+2\tau^{\frac{1}{2}})] = 0.
\end{aligned}$$

但 (7) 所给的两个解并不常是实的, 为了研究实解答, 我们用以下的两个实解:

$$P_n(\tau) = \frac{1}{2}[(1+\tau^{\frac{1}{2}})^n + (1-\tau^{\frac{1}{2}})^n] \tag{8}$$

及

$$|\tau|^{\frac{1}{2}}Q_n(\tau), \tag{9}$$

此处 $Q_n(\tau) = \dfrac{1}{2\tau^{1/2}}((1+\tau^{\frac{1}{2}})^n - (1-\tau^{\frac{1}{2}})^n)$, 我们取

$$\tau^{\frac{1}{2}} = \begin{cases} |\tau|^{\frac{1}{2}}, & 若\ \tau>0, \\ i|\tau|^{\frac{1}{2}}, & 若\ \tau<0. \end{cases}$$

所以方程 (D) 有如下形式的一些解:

$$\frac{P_n(\tau)}{\rho^n}\cos n\theta, \quad \frac{P_n(\tau)}{\rho^n}\sin n\theta, \quad |\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}\cos n\theta, \quad |\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}\sin n\theta,$$

此处 $n=1,2,3,\cdots$.

当 $n=0$ 时, 以上的处理方法必须补充, 由 (1) 得

$$\rho^2(1-\rho^2)\varphi''(\rho) + \rho(1-2\rho^2)\varphi'(\rho) = 0, \quad \psi''(\theta) = 0.$$

前式即

$$\frac{\varphi''(\theta)}{\varphi'(\rho)} = \frac{2\rho^2-1}{\rho(1-\rho^2)} = -\frac{1}{\rho} - \frac{1}{2}\frac{1}{1+\rho} + \frac{1}{2}\frac{1}{1-\rho},$$

所以

$$\varphi'(\rho) = -C_1\frac{|\tau|^{\frac{1}{2}}}{\rho\tau}.$$

即

$$\varphi(\rho) = \begin{cases} C_1\log\dfrac{1+\sqrt{1-\rho^2}}{\rho} + C_2, & \text{当 } \rho < 1, \\ C_1\text{arc cos}\,\dfrac{1}{\rho} + C_2, & \text{当 } \rho > 1 \end{cases}$$

(假定了 $\varphi(\rho)$ 在 $\rho = 1$ 连续), 同时 $\psi(\theta) = C_3\theta + C_4$. 由于 $u(\rho,\theta)$ 是 θ 的以 2π 为周期的函数, 所以 $C_3 = 0$. 命

$$\sigma(\rho) = \begin{cases} \log\dfrac{1+\sqrt{1-\rho^2}}{\rho}, & \text{当 } \rho < 1, \\ \text{arc cos}\,\dfrac{1}{\rho}, & \text{当 } \rho > 1. \end{cases}$$

如此则

$$\lim_{\rho\to 1\pm 0}\frac{\sigma(\rho)}{|\tau|^{\frac{1}{2}}} = 1.$$

这建议了微分方程 (D) 有以下形式的解答

$$\begin{aligned} u(\rho,\theta) =& \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta) \times \frac{P_n(\tau)}{\rho^n} \\ &+ c_0\sigma(\rho) + |\tau|^{\frac{1}{2}} \times \sum_{n=1}^{\infty}(c_n\cos n\theta + d_n\sin n\theta)\frac{Q_n(\tau)}{\rho^n}. \end{aligned} \tag{F}$$

现在暂不讨论 (F) 的收敛问题及其是否适合于 (D) 的问题, 而先虚瞰一下, 由 (F) 建议出些什么来.

首先我们应当肯定讨论那一类的函数, 由 (F) 的形式可以看出, 当 $\rho = 1$ 时, 我们必须减弱条件, 也就是我们不能假定 $u(\rho,\theta)$ 的微商在 $\rho = 1$ 处存在, 但应当假定

$$\lim_{\rho\to 1-0}\frac{u(\rho,\theta)-u(1,\theta)}{|\tau|^{\frac{1}{2}}} = \lim_{\rho\to 1+0}\frac{u(\rho,\theta)-u(1,\theta)}{|\tau|^{\frac{1}{2}}} \tag{10}$$

存在.

切实些说, 我们所研究的函数类如下:

给一个区域 D, 其中包有一段单位圆, 当 $\rho \neq 1$ 时, 函数 $u(\rho,\theta)$ 有二阶偏微商, 但当 $\rho = 1$ 时, 我们假定适合于 (10).

如果我们限定所考虑的函数是实解析的, 则 (F) 的第二部分不存在, 因而也没有必要假定条件 (10) 了.

7.6　问题的提出 (虚瞰)

首先, 在单位圆 (变型线) 上级数 (F) 的情况, 我们有

$$u(1,\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$
$$=\varphi(\theta)\qquad(\text{定义}).\tag{1}$$

这建议了：如果仅仅给了单位圆上的数值 (即 $u(1,\theta)=\varphi(\theta)$). 方程 (D) 的解答并不唯一, 因为对任何的 c_n, d_n, (F) 在单位圆上都有相同的值, 而且都是 (D) 的解答.

这也建议了, 我们应当考虑适合以下条件的函数：极限

$$\lim_{\rho\to1}\frac{u(\rho,\theta)-\varphi(\theta)}{|\tau|^{\frac{1}{2}}}\tag{2}$$

存在, 即

$$\lim_{\rho\to1}\frac{u(\rho,\theta)-\varphi(\theta)}{|\tau|^{\frac{1}{2}}}=\lim_{\tau\to0}\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta)Q_n(\tau)+c_0\lim_{\rho\to\pm0}\frac{\sigma(\rho)}{|\tau|^{\frac{1}{2}}}$$
$$=\sum_{n=1}^{\infty}n(c_n\cos n\theta+d_n\sin n\theta)+c_0=\chi(\theta)\quad(\text{定义}).$$

问题 I　给了两个函数 $\varphi(\theta)$ 与 $\chi(\theta)$, 以 2π 为周期, 在什么条件下, 有一个且仅有一个函数 $u(\rho,\theta)$ 适合于 (D)(单位圆除外, 但在单位圆上连续) 而且

$$u(\rho,\theta)|_{\rho=1}=\varphi(\theta)\tag{3}$$

及

$$\lim_{\rho\to1}\frac{u(\rho,\theta)-\varphi(\theta)}{|\tau|^{1/2}}=\chi(\theta).\tag{4}$$

如果 $\varphi(\theta)$ 与 $\chi(\theta)$ 的 Fourier 级数各为

$$\varphi(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\tag{5}$$

及

$$\chi(\theta)=\frac{1}{2}\gamma_0+\sum_{n=1}^{\infty}(\gamma_n\cos n\theta+\delta_n\sin n\theta),\tag{6}$$

则解答就是

$$\begin{aligned}u(\rho,\theta)=&\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\frac{P_n(\tau)}{\rho^n}\\&+\frac{1}{2}\gamma_0\sigma(\rho)+|\tau|^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{1}{n}(\gamma_n\cos n\theta+\delta_n\sin n\theta)\frac{Q_n(\tau)}{\rho^n}.\end{aligned}\tag{7}$$

如果假定解答是实解析的, 则条件 (3) 就有希望唯一决定 (D) 的解了.

其次, 考虑在特征线上的情况, 取特征线 $x=1$, 即

$$\rho\cos\theta=1,\quad |\theta|\leqslant\frac{\pi}{2},$$

现在

$$\begin{aligned}\frac{P_n(\tau)}{\rho^n}&=\frac{1}{2}\left[\left(\frac{1}{\rho}+\frac{\tau^{\frac{1}{2}}}{\rho}\right)^n+\left(\frac{1}{\rho}-\frac{\tau^{\frac{1}{2}}}{\rho}\right)^n\right]\\&=\frac{1}{2}((\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n)\\&=\cos n\theta.\end{aligned}$$

同样, 我们有

$$|\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}=\sin n\theta$$

及

$$\sigma(\rho)=\theta.$$

因此得到

$$\begin{aligned}u\left(\frac{1}{\cos\theta},\theta\right)=&\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\cos n\theta\\&+c_0\theta+\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta)\sin n\theta\\=&\frac{1}{2}a_0+\frac{1}{2}\sum_{n=1}^{\infty}[a_n(1+\cos 2n\theta)+b_n\sin 2n\theta]\\&+c_0\theta+\frac{1}{2}\sum_{n=1}^{\infty}[c_n\sin 2n\theta+d_n(1-\cos 2n\theta)]\\=&\frac{1}{2}\left[a_0+\sum_{n=1}^{\infty}(a_n+d_n)\right]+c_0\theta\\&+\frac{1}{2}\sum_{n=1}^{\infty}[(a_n-d_n)\cos 2n\theta+(b_n+c_n)\sin 2n\theta]\end{aligned}$$

$$=\tau(\theta) \quad (定义). \tag{8}$$

由

$$u(1,0)=\varphi(0)=\tau(0), \tag{9}$$

可以推出以下的问题.

问题 II　给了两个函数 $\varphi(\theta)$ 与 $\tau(\theta)$, $\varphi(\theta)$ 与 $\tau(\theta)$ 各以 2π 及 π 为周期, 而且 $\varphi(0)=\tau(0)$. 在怎样的条件下, 有函数 $u(\rho,\theta)$ 存在适合于 (D)(单位圆上仍有如前的假定) 而且

$$u(\rho,\theta)|_{\rho=1}=\varphi(\theta), \quad u(\rho,\theta)|_{x=1}=\tau(\theta).$$

如果 $\varphi(\theta)$ 的 Fourier 展开式 (6) 及 $\tau(\theta)$ 有 Fourier 展开式

$$\tau(\theta)=\frac{1}{2}\alpha_0+\gamma_0\theta+\sum_{n=1}^{\infty}(\alpha_n\cos 2n\theta+\beta_n\sin 2n\theta) \tag{10}$$

及

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n=\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}\alpha_n,$$

则解答应当是

$$\begin{aligned}u(\rho,\theta)=&\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\frac{P_n(\tau)}{\rho^n}\\&+\gamma_0\sigma(\rho)+|\tau|^{\frac{1}{2}}\sum_{n=1}^{\infty}[(2\beta_n-b_n)\cos n\theta-(2\alpha_n-a_n)\sin n\theta]\frac{Q_n(\tau)}{\rho^n}.\end{aligned} \tag{11}$$

再考虑圆内的情况, 我们引进新变数

$$\lambda=\frac{1}{\rho}-\sqrt{\frac{1}{\rho^2}-1}=\frac{1-\tau^{\frac{1}{2}}}{\rho}=\frac{\rho}{1+\tau^{\frac{1}{2}}}. \tag{12}$$

在这变换下, 把微分方程 (D) 变为

$$\frac{\partial^2 u}{\partial\lambda^2}+\frac{1}{\lambda}\frac{\partial u}{\partial\lambda}+\frac{1}{\lambda^2}\frac{\partial^2 u}{\partial\theta^2}=0.$$

这恰好就是 Laplace 方程的极坐标形式, 由于

$$\frac{d\lambda}{d\rho}=\frac{1-\sqrt{1-\rho^2}}{\rho^2\sqrt{1-\rho^2}}\geqslant 0,$$

当 ρ 由 0 变到 1 时, λ 也由 0 变到 1, 并且对应是 1-1 的, 这样便有

$$\frac{P_n(\tau)}{\rho^n}=\frac{1}{2\rho^n}[(1+\tau^{\frac{1}{2}})^n+(1-\tau^{\frac{1}{2}})^n]$$

$$=\frac{1}{2}\left[\left(\frac{1+\tau^{\frac{1}{2}}}{\rho}\right)^n+\left(\frac{1-\tau^{\frac{1}{2}}}{\rho}\right)^n\right]$$
$$=\frac{\lambda^n+\lambda^{-n}}{2}, |\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}$$
$$=\frac{1}{2}\left[\left(\frac{1+\tau^{\frac{1}{2}}}{\rho}\right)^n-\left(\frac{1-\tau^{\frac{1}{2}}}{\rho}\right)^n\right]=\frac{-\lambda^n+\lambda^{-n}}{2},$$

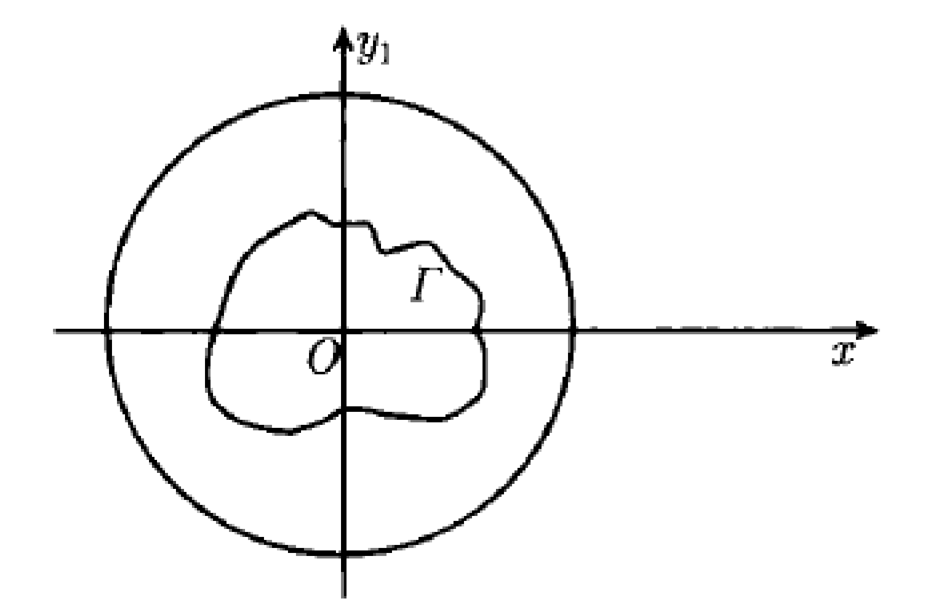
及 $\sigma(\rho)=\log\lambda$. 所以

$$\begin{aligned}u(\rho,\theta)=&\frac{1}{2}a_0+\frac{1}{2}\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)(\lambda^n+\lambda^{-n})\\&+c_0\log\lambda+\frac{1}{2}\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta)(-\lambda^n+\lambda^{-n})\\=&\frac{1}{2}a_0+\frac{1}{2}\sum_{n=1}^{\infty}[(a_n+c_n)\cos n\theta+(b_n+d_n)\sin n\theta]\lambda^{-n}\\&+c_0\log\lambda+\frac{1}{2}\sum_{n=1}^{\infty}[(a_n-c_n)\cos n\theta+(b_n-d_n)\sin n\theta]\lambda^n\\=&U(\lambda,\theta),\end{aligned}\tag{13}$$

这里 $U(\lambda,\theta)$ 是一个在环状域内的普通的单值调和函数, 这也就是一个环状域的解析函数的实数部分, 因而建议了

问题 III 在变型线 (单位圆) 上及圆内一闭曲线 Γ 上给了 $u(\rho,\theta)$ 的函数值:

$$u(\rho,\theta)|_{\rho=1}=\varphi(\theta),$$
$$u(\rho,\theta)|_{\Gamma}=\psi(\theta),\tag{14}$$



则 (D) 的解是存在而且唯一的.

特别 Γ 是同心圆有

问题 IV 给了

$$u(\rho,\theta)|_{\rho=1}=\Phi(\theta),$$
$$u(\rho,\theta)|_{\rho=\rho_0}=\Psi(\theta),\quad 0<\rho_0<1,$$

求 $u(\rho,\theta)$.

有以下处理方法.

在极坐标 (λ,θ) 的平面上考虑问题, 假定 Γ 是以 λ_0 为半径, 原点为中心的圆. 在 $0<\lambda_0<\lambda<1$ 中考虑函数

$$\begin{aligned}W(\lambda,\theta)=&\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\lambda^2}{1-2\lambda\cos(\theta-\psi)+\lambda^2}\Phi(\psi)d\psi\\&+\frac{1}{2\pi}\int_0^{2\pi}\frac{\lambda^2-\lambda_0^2}{\lambda^2-2\lambda\lambda_0\cos(\theta-\psi)+\lambda_0^2}\Psi(\psi)d\psi+\gamma\log\lambda.\end{aligned}\tag{15}$$

此处 $\Phi(\psi)$ 与 $\Psi(\psi)$ 可以有以下的 Fourier 级数表出的函数

$$\begin{cases}\Phi(\theta)=\dfrac{1}{2}\alpha_0+\displaystyle\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta),\\\Psi(\theta)=\dfrac{1}{2}\gamma_0+\displaystyle\sum_{n=1}^{\infty}(\gamma_n\cos n\theta+\delta_n\sin n\theta).\end{cases}\tag{16}$$

把 Poisson 核展开逐项求积分可得

$$\begin{aligned}W(\lambda,\theta)=&\frac{1}{2}(\alpha_0+\gamma_0)+\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\lambda^n\\&+\gamma\log\lambda+\sum_{n=1}^{\infty}(\gamma_n\cos n\theta+\delta_n\sin n\theta)\left(\frac{\lambda_0}{\lambda}\right)^n.\end{aligned}\tag{17}$$

当 $\lambda\to1$ 及 $\lambda\to\lambda_0$, 则各得

$$W(\lambda,\theta)|_{\lambda=1}=\frac{1}{2}(\alpha_0+\gamma_0)+\sum_{n=1}^{\infty}[(\alpha_0+\lambda_0^n\gamma_n)\cos n\theta+(\beta_n+\lambda_0^n\delta_n)\sin n\theta].\tag{18}$$

$$\begin{aligned}W(\lambda,\theta)|_{\lambda=\lambda_0}=&\frac{1}{2}(\alpha_0+\gamma_0)+\gamma\log\lambda_0\\&+\sum_{n=1}^{\infty}[(\alpha_n\lambda_0^n+\gamma_n)\cos n\theta+(\beta_n\lambda_0^n+\delta_n)\sin n\theta].\end{aligned}\tag{19}$$

把 (17) 换为 (ρ,θ) 符号, 则得函数

$$\begin{aligned}u(\rho,\theta)=&W(\lambda,\theta)\\=&\frac{1}{2}(\alpha_0+\gamma_0)+\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\left[\frac{P_n(\tau)}{\rho^n}+|\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}\right]\\&+\gamma\sigma(\rho)+\sum_{n=1}^{\infty}(\gamma_n\cos n\theta+\delta_n\sin n\theta)\lambda_0^n\left[\frac{P_n(\tau)}{\rho^n}-|\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}\right]\\=&\frac{1}{2}(\alpha_0+\gamma_0)+\sum_{n=1}^{\infty}[(\alpha_n+\gamma_n\lambda_0^n)\cos n\theta+(\beta_n+\delta_n\lambda_0^n)\sin n\theta]\frac{P_n(\tau)}{\rho^n}\end{aligned}$$

$$+\gamma\sigma(\rho)+|\tau|^{\frac{1}{2}}\sum_{n=1}^{\infty}[(\alpha_n-\gamma_n\lambda_0^n)\cos n\theta+(\beta_n-\delta_n\lambda_0^n)\sin n\theta]\frac{Q_n(\tau)}{\rho^n}. \quad (20)$$

如果 (14) 所给的函数的 Fourier 级数是

$$\varphi(\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),$$
$$\psi(\theta)=\frac{1}{2}c_0+\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta). \quad (21)$$

与 (18), (19) 比较便得

$$\alpha_0+\gamma_0=a_0,\quad \alpha_n+\lambda_0^n\gamma_n=a_n,\quad \beta_n+\lambda_0^n\delta_n=b_n,$$
$$\alpha_0+\gamma_0+2\gamma\log\lambda_0=c_0,\quad \alpha_n\lambda_0^n+\gamma_n=c_n,$$
$$\beta_n\lambda_0^n+\delta_n=d_n.$$

代入 (20), 我们可以希望问题 IV 的解的形式是

$$\begin{aligned}u(\rho,\theta)=&\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\frac{P_n(\tau)}{\rho^n}\\&+\frac{c_0-a_0}{2\log\lambda_0}\sigma(\rho)+|\tau|^{\frac{1}{2}}\sum_{n=1}^{\infty}\left[\left(\frac{1+\lambda_0^{2n}}{1-\lambda_0^{2n}}a_n-\frac{2\lambda_0^n}{1-\lambda_0^{2n}}c_n\right)\cos n\theta\right.\\&\left.+\left(\frac{1+\lambda_0^{2n}}{1-\lambda_0^{2n}}b_n-\frac{2\lambda_0^n}{1-\lambda_0^{2n}}d_n\right)\sin n\theta\right]\frac{Q_n(\tau)}{\rho^n}\\=&\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\frac{P_n(\tau)}{\rho^n}+\frac{1}{2}(c_0-a_0)\frac{\sigma(\rho)}{\sigma(\rho_0)}\\&+\left|\frac{\tau}{\tau_0}\right|^{\frac{1}{2}}\sum_{n=1}^{\infty}\left[\left(\frac{-P_n(\tau_0)a_n+c_n\rho_0^n}{Q_n(\tau_0)}\right)\cos n\theta\right.\\&\left.+\left(\frac{-P_n(\tau_0)b_n+d_n\rho_0^n}{Q_n(\tau_0)}\right)\sin n\theta\right]\frac{Q_n(\tau)}{\rho^n},\end{aligned}$$

此处 ρ_0,τ_0 是当 $\lambda=\lambda_0$ 时 ρ 与 τ 的值.

问题 V(Tricomi) 命 Γ 的圆内的一条闭曲线, 假定在 Γ 上及一条特征线 (例如 $x=1$) 上给了函数值, 则 (D) 的解答是唯一的.

我们还是取 Γ 是一个同心圆的情况, 假定

$$u(\rho,\theta)|_{\Gamma}=\psi(\theta),\quad u(\rho,\theta)|_{x=1}=\tau(\theta).$$

为了解决这问题, 我们把问题 II 及 IV 的解等同起来, 先假定有

$$u(\rho,\theta)|_{\rho=1}=\varphi(\theta).$$

比较 (11) 与 (22) 可知

$$\gamma_0 = \frac{c_0 - a_0}{2\log\lambda_0}, \quad 2\beta_n - b_n = \frac{(1+\lambda_0^{2n})a_n - 2\lambda_0^n c_n}{1-\lambda_0^{2n}},$$
$$-2\alpha_n + a_n = \frac{(1+\lambda_0^{2n})b_n - 2\lambda_0^n d_n}{1-\lambda_0^{2n}}.$$

由此解出 a_0, a_n, b_n, 代入 (11) 式即得所求.

问题 VI　假定了

$$u(\rho,\theta)|_{\rho=\rho_0} = \psi(\theta), \quad u(\rho,\theta)|_{x=1} = \tau(\theta)$$

而求解.

在 (F) 中代入以上的条件得出

$$u(\rho,\theta)|_{x=1} = \frac{1}{2}\left[a_0 + \sum_{n=1}^{\infty}(a_n + d_n)\right] + c_0\theta$$
$$+ \frac{1}{2}\sum_{n=1}^{\infty}[(a_n - d_n)\cos 2n\theta + (b_n + c_n)\sin 2n\theta]$$

及

$$u(\rho,\theta)|_{\rho=\rho_0} = \frac{1}{2}a_0 + c_0\sigma(\rho_0) + \sum_{n=1}^{\infty}\left\{\left[a_n\frac{P_n(\tau_0)}{\rho_0^n} + |\tau_0|^{\frac{1}{2}}c_n\frac{Q_n(\tau_0)}{\rho_0^n}\right]\cos n\theta\right.$$
$$\left.+\left[b_n\frac{P_n(\tau_0)}{\rho_0^n} + |\tau_0|^{\frac{1}{2}}d_n\frac{Q_n(\tau_0)}{\rho_0^n}\right]\sin n\theta\right\}.$$

与 (10) 及 (21)(把其中的 c, d 改为 c', d') 相比较得

$$\alpha_0 = a_0 + \sum_{n=1}^{\infty}(a_n + d_n), \quad \gamma_0 = c_0, \ \alpha_n = \frac{1}{2}(a_n - d_n),$$
$$\beta_n = \frac{1}{2}(b_n + c_n), \quad c_0' = a_0 + 2c_0\sigma(\rho_0),$$
$$c_n' = a_n\frac{P_n(\tau_0)}{\rho_0^n} + |\tau|^{\frac{1}{2}}c_n\frac{Q_n(\tau_0)}{\rho_0^n},$$
$$d_n' = b_n\frac{P_n(\tau_0)}{\rho_0^n} + |\tau|^{\frac{1}{2}}d_n\frac{Q_n(\tau_0)}{\rho_0^n}.$$

7.7　级数的收敛性

现在我们考虑级数

$$u(\rho,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)\times\frac{P_n(\tau)}{\rho^n}$$

$$+c_0\sigma(\rho)+|\tau|^{\frac{1}{2}}\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta)\frac{Q_n(\tau)}{\rho^n} \tag{F}$$

的收敛性及是否适合于方程 (D) 的问题.

我们先考虑单位圆外的情况, 即 $\rho\geqslant 1$, 命

$$\rho=\frac{1}{\cos\eta},\quad 0\leqslant\eta<\frac{\pi}{2},$$

则

$$\frac{P_n(\tau)}{\rho^n}=\frac{1}{2}\left(\left(\frac{1+\tau^{\frac{1}{2}}}{\rho}\right)^n+\left(\frac{1-\tau^{\frac{1}{2}}}{\rho}\right)^n\right)=\cos n\eta,$$

$$|\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}=\frac{1}{2i}\left(\left(\frac{1+\tau^{\frac{1}{2}}}{\rho}\right)^n-\left(\frac{1-\tau^{\frac{1}{2}}}{\rho}\right)^n\right)=\sin n\eta,$$

$$\sigma(\rho)=\eta.$$

如此, 级数 (F) 变为

$$\begin{aligned}u(\rho,\theta)=&\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\cos n\eta\\&+c_0\eta+\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta)\sin n\eta.\end{aligned} \tag{1}$$

由于

$$\begin{aligned}\frac{\partial^2u(\rho,\theta)}{\partial\theta^2}=&-\sum_{n=1}^{\infty}n^2(a_n\cos n\theta+b_n\sin n\theta)\cos n\eta\\&-\sum_{n=1}^{\infty}n^2(c_n\cos n\theta+d_n\sin n\theta)\sin n\eta.\end{aligned} \tag{2}$$

及

$$\begin{aligned}\frac{\partial^2u(\rho,\theta)}{\partial\rho^2}=&\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\Bigg(-n^2\cos n\eta\left(\frac{d\eta}{d\rho}\right)^2\\&-n\sin n\eta\frac{d\eta}{d\rho}\Bigg)+\sum_{n=1}^{\infty}(c_n\cos n\theta+d_n\sin n\theta)\\&\times\left[-n^2\sin n\eta\left(\frac{d\eta}{d\rho}\right)^2+n\cos n\eta\left(\frac{d\eta}{d\rho}\right)\right],\end{aligned} \tag{3}$$

所以只要假定了

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|+|c_n|+|d_n|)n^2<\infty, \tag{4}$$

则级数 (1), (2), (3) 都一致收敛, 因此, (F) 在单位圆外适合于方程式 (D).

条件 (4) 当然可由假定 $\varphi(\theta)$ 有四阶连续微商, $\chi(\theta)$ 有三阶连续微商推出之. 当然也可由 $\varphi(\theta)$ 与 $\tau(\theta)$ 都有四阶连续微商推出之 (或用其他 Fourier 级数论中常见的更弱的条件).

再论圆内的情况: 我们将证明两点:

1° 如果级数 (F) 当 $\rho=\rho_0$, $\theta=\theta_0$ 收敛, 则当 $\rho>\rho_0$, $\theta=\theta_0$ 时也收敛.

2° 如果它当 $\rho=\rho_0$ 及一个测度为正的 θ 集合上收敛, 则当 $\rho\geqslant\rho_0$ 时收敛 (并且在这区域中任何一个有限区域一致收敛).

1° 的证明十分容易, 因为 $P_n(\tau)/\rho^n$, $Q_n(\tau)/\rho^n$ 是 ρ 的递减函数, 可以从

$$\varlimsup_{n\to\infty}\left|(a_n\cos n\theta+b_n\sin n\theta)\frac{P_n(\tau_0)}{\rho_0^n}\right|^{\frac{1}{n}}\leqslant 1$$

及

$$\varlimsup_{n\to\infty}\left|(c_n\cos n\theta+d_n\sin n\theta)\frac{Q_n(\tau_0)}{\rho_0^n}\right|^{\frac{1}{n}}\leqslant 1$$

立刻推得 $\rho>\rho_0$ 时, 有 $\mu<1$ 存在使

$$\varlimsup_{n\to\infty}\left|(a_n\cos n\theta+b_n\sin n\theta)\frac{P_n(\tau)}{\rho^n}\right|^{\frac{1}{n}}\leqslant \mu$$

及

$$\varlimsup_{n\to\infty}\left|(c_n\cos n\theta+d_n\sin n\theta)\frac{Q_n(\tau)}{\rho^n}\right|^{\frac{1}{n}}\leqslant \mu.$$

要证明 2°, 我们需用引理: 如果有一正测度的点集 θ, 其上任一点 θ 有

$$\varlimsup_{n\to\infty}|a_n\cos n\theta+b_n\sin n\theta|^{\frac{1}{n}}=\nu,$$

则

$$\varlimsup_{n\to\infty}|a_n^2+b_n^2|^{\frac{1}{2n}}=\nu.$$

这引理是已知的 (或可以从数论中的一致分布概念推出)(Лузин, Steinhaus 见 Zygmund, Trigonometrial Sereis, p.131 及 p.269).

有了这引理立刻可用前法推出 2° 来, 并可知推出

$$a_n,\ b_n,\ c_n,\ d_n=O(\rho_0^n).$$

这条件如果适合, 则条件 (4) 当然成立.

总之, 如果在单位圆内以原点为中心的一个圆上, 有一正测度的点集在它上 (下) 收敛, 则在此圆之外无处不收敛, 并且适合于微分方程 (D).

7.8 圆内无奇点的函数 (对应于全纯函数)

问题 在一条特征线上给定了函数值, 定出一个处处都适合 (D) 的函数来. 这样的函数是否有? 是否唯一?

由于特征线成一可递集，因此研究哪一条特征线反正都一样. 我们不妨就取 $x=1$，即

$$\rho=\frac{1}{\cos\theta},\quad |\theta|\leqslant\frac{\pi}{2}.$$

于是问题更确切的叙述是，假定

$$u(\rho,\theta)|_{x=1}=\tau(\theta),$$

求出适合于 (D) 的 $u(\rho,\theta)$.

(1) 存在性. 由于客观需要, 我们假定 $\tau(\theta)$ 有二阶微商 (如果 $u(\rho,\theta)$ 对 θ 没有二阶偏微商, 那我们只可以作为广义解来讨论) 并且假定 $\tau(\theta)$ 是以 π 为周期的函数. 命

$$\tau(\theta)=\frac{1}{2}p_0+\sum_{n=1}^{\infty}(p_n\cos 2n\theta+q_n\sin 2n\theta)$$

是 $\tau(\theta)$ 的 Fourier 展开式, 级数

$$\sum_{n=1}^{\infty}(|p_n|+|q_n|)$$

显然收敛.

结论：命

$$\alpha_n=p_n+q_n,\quad \beta_n=q_n-p_n,\quad \alpha_0=p_0-2\sum_{n=1}^{\infty}q_n,$$

则

$$u(\rho,\theta)=\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta)\frac{P_n(\tau)+|\tau|^{\frac{1}{2}}Q_n(\tau)}{\rho^n}\tag{1}$$

就适合我们的要求.

先看在特征线上

$$\frac{P_n(\tau)}{\rho^n}=\frac{1}{2}\left[\left(\frac{1}{\rho}+i\sqrt{1-\frac{1}{\rho^2}}\right)^n+\left(\frac{1}{\rho}-i\sqrt{1-\frac{1}{\rho^2}}\right)^n\right]=\cos n\theta,$$

$$|\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}=\sin n\theta.$$

因此

$$\begin{aligned}u(\rho,\theta)|_{x=1} =&\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta)(\cos n\theta+\sin n\theta)\\ =&\frac{1}{2}\alpha_0+\frac{1}{2}\sum_{n=1}^{\infty}[\alpha_n(\cos 2n\theta+1+\sin 2n\theta)\\ &+\beta_n(\sin 2n\theta-\cos 2n\theta+1)]\\ =&\frac{1}{2}p_0+\sum_{n=1}^{\infty}(p_n\cos 2n\theta+q_n\sin 2n\theta)=\tau(\theta).\end{aligned}$$

其次在圆外, (1) 处处收敛, 而且适合于 (D). 在单位圆外命

$$\rho=\frac{1}{\cos\eta},\quad 0\leqslant\eta<\frac{\pi}{2},$$

则

$$\frac{P_n(\tau)}{\rho^n}=\cos n\eta,\quad |\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n}=\sin n\eta.$$

因而 (1) 变为

$$u(\rho,\theta)=\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos n\theta+\beta_n\sin n\theta)(\cos n\eta+\sin n\eta).$$

由于 $\sum\limits_{n=1}^{\infty}(|\alpha_n|+|\beta_n|)<\infty$, 这级数的收敛没有问题.

如果假定了

$$\sum_{n=1}^{\infty}n^2(|p_n|+|q_n|)<\infty, \tag{2}$$

则

$$\frac{\partial^2u(\rho,\theta)}{\partial\theta^2}=-\sum_{n=1}^{\infty}n^2(\alpha_n\cos n\theta+\beta_n\sin n\theta)(\cos n\eta+\sin n\eta)$$

及

$$\begin{aligned}\rho(\rho^2-1)^{\frac{1}{2}}\frac{\partial u(\rho,\theta)}{\partial\rho}&=\rho(\rho^2-1)^{\frac{1}{2}}\frac{\partial u(\rho,\theta)}{\partial\eta}\frac{\partial\eta}{\partial\rho}=\frac{\partial u(\rho,\theta)}{\partial\eta}\\ &=\sum_{n=1}^{\infty}n(\alpha_n\cos n\theta+\beta_n\sin n\theta)(-\sin n\eta+\cos n\eta).\end{aligned}$$

由此可知 (1) 适合于方程式 (D), 当然我们要添一些假定: 例如 $\tau(\theta)$ 有四阶连续微商. 但如果不适合这个假定, 我们也不妨把 (1) 看为方程 (D) 在双曲区的广义解.

最后, 在单位圆内, 我们引进新变数

$$\lambda = \frac{1}{\rho} - \sqrt{\frac{1}{\rho^2} - 1} = \frac{1 - \tau^{\frac{1}{2}}}{\rho} = \frac{\rho}{1 + \tau^{\frac{1}{2}}}.$$

由于

$$\frac{d\lambda}{d\rho} = \frac{1 - \sqrt{1 - \rho^2}}{\rho^2\sqrt{1 - \rho^2}} \geqslant 0,$$

当 ρ 由 0 变到 1 时, λ 也单调上升的由 0 变到 1. 这样便有

$$\frac{P_n(\tau)}{\rho^n} = \frac{1}{2}\left(\left(\frac{1 + \tau^{\frac{1}{2}}}{\rho}\right)^n + \left(\frac{1 - \tau^{\frac{1}{2}}}{\rho}\right)^2\right) = \frac{1}{2}(\lambda^n + \lambda^{-n}),$$

$$|\tau|^{\frac{1}{2}}\frac{Q_n(\tau)}{\rho^n} = \frac{1}{2}(-\lambda^n + \lambda^{-n}).$$

因此级数 (1) 变为

$$u(\rho, \theta) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos n\theta + \beta_n \sin n\theta)\lambda^{-n},$$

它是 (λ^{-n}, θ) 的调和函数, 并且圆内处处收敛, 因而在圆内 (1) 也适合于 (D).

(2) 唯一性. 我们证明, 只有 $u(\rho, \theta) \equiv 0$ 才能适合于

$$u(\rho, \theta)|_{x=1} = 0,$$

而且单位圆内无奇点.

在圆外有通解

$$u(\rho, \theta) = F_1\left(\theta + \cos^{-1}\frac{1}{\rho}\right) + F_2\left(\theta - \cos^{-1}\frac{1}{\rho}\right),$$

$$F_2(0) = 0,$$

即得

$$u(\rho, \theta)|_{x=1} = F_1(2\theta) = 0.$$

因此在圆外,

$$u(\rho, \theta) = F_2\left(\theta - \cos^{-1}\frac{1}{\rho}\right), \quad F_2(0) = 0. \tag{3}$$

由方程 (D) 可知

$$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=1} = \left.\frac{\partial^2 u}{\partial \theta^2}\right|_{\rho=1} = \left.\frac{\partial}{\partial \rho}F_2\left(\theta - \cos^{-1}\frac{1}{\rho}\right)\right|_{\rho=1}$$

$$= F_2'\left(\theta - \cos^{-1}\frac{1}{\rho}\right)\frac{1}{\rho\sqrt{\rho^2-1}}\bigg|_{\rho=1}.$$

当 $\rho = 1$ 时分母为 0, 因此 $F_2' = 0$. 命

$$U(\lambda,\theta) = u(\rho,\theta),$$

则得

$$\lim_{\rho\to 1-0}\frac{u(\rho,\theta)-u(1,\theta)}{|\tau|^{\frac{1}{2}}} = \lim_{\lambda\to 1-0}\frac{U(\lambda,\theta)-U(1,\theta)}{1-\lambda} = -\frac{\partial U}{\partial\lambda}\bigg|_{\lambda=1},$$

也就是

$$\frac{\partial U}{\partial\lambda}\bigg|_{\lambda=1} = -\frac{\partial U}{\partial\theta}\bigg|_{\lambda=1}$$

假定 U 是复变数解析函数 $f(z)$ 的实数部分, 而 V 乃其虚数部分, 则

$$f(z) = U + iV$$

把单位圆 $|z| = 1$ 变为一直线

$$U + V = 0.$$

运用 Schwarz 对称定理, 在某些必须添加的条件下, 函数 $f(z)$ 的解析性可以扩展到全平面 (包括无穷远点), 因而证出 $V = 0$.

7.9 圆内有对数奇点的函数

先从

$$\log\lambda$$

出发. 它是一个圆内以原点为对数奇点的调和函数. 换为 (ρ,θ) 符号, 则得方程 (D) 的一个基本解

$$\sigma(\rho) = \begin{cases} -\log\left(\dfrac{1}{\rho} - \sqrt{\dfrac{1}{\rho^2}-1}\right), & \text{当 } \rho \leqslant 1, \\ \arccos\dfrac{1}{\rho}, & \text{当 } \rho \geqslant 1 \end{cases}$$

或

$$\sigma(x,y) = \begin{cases} -\log\left(\dfrac{1-\sqrt{1-x^2-y^2}}{\sqrt{x^2+y^2}}\right), & \text{当 } x^2+y^2 \leqslant 1, \\ \arccos\dfrac{1}{\sqrt{x^2+y^2}}, & \text{当 } x^2+y^2 \geqslant 1. \end{cases}$$

现在利用群的作用, 命

$$x = f(x', y', a, b), \quad y = g(x', y', a, b)$$

代表一个把 $(a,\ b)$ 变为原点的变换, 这样

$$\sigma(x, y) = \sigma(f(x', y'; a, b), g(x', y'; a, b)) = \sigma_{a,b}(x', y').$$

由于偏微分方程的不变性可知 $\sigma_{a,b}(x, y)$ 依然是 (D) 的解答.

命 $\mu(a, b)$ 是任一分布函数, 则

$$F(x, y) = \iint\limits_{a^2+b^2\leqslant 1} \sigma_{a,b}(x, y) d\mu(a, b)$$

依然是方程 (D) 的解答.

问题 $F(1, y)$ 是怎样的函数类, 也便是从怎样的函数 $\{\varphi(y)\}$ 类中, 对每一个 $\varphi(y)$, 我们可以找到一个 $\mu(a, b)$ 使

$$\varphi(y) = \iint\limits_{a^2+b^2\leqslant 1} \sigma_{a,b}(1, y) d\mu(a, b).$$

更具体些:

$$x_1 = (x\cos\alpha + y\sin\alpha - \mu)/(1 - \mu x\cos\alpha - \mu y\sin\alpha),$$
$$y_1 = \sqrt{1-\mu^2}(-x\sin\alpha + y\cos\alpha)/(1 - \mu x\cos\alpha - \mu y\sin\alpha)$$

是一个 Γ 内的变形, 它把圆内的一点 $(\mu\cos\alpha, \mu\sin\alpha)$ 变为 (0,0). 因而函数

$$\begin{aligned}
&\sigma\mu\cos\alpha, \mu\sin\alpha(x, y)\\
=&\sigma\left(\frac{\sqrt{(x\cos\alpha + y\sin\alpha - \mu)^2 + (1-\mu^2)(-x\sin\alpha + y\cos\alpha)^2}}{1 - \mu x\cos\alpha - \mu y\sin\alpha}\right)\\
=&\sigma\left(\frac{\sqrt{(1-\mu^2)(x^2+y^2-1) + (1-\mu(x\cos\alpha + y\sin\alpha))^2}}{1 - \mu(x\cos\alpha + y\sin\alpha)}\right)\\
=&\sigma\left(\frac{\sqrt{(1-\mu^2)(\rho^2-1) + (1-\mu\rho\cos(\alpha-\theta))^2}}{1 - \mu\rho\cos(\alpha-\theta)}\right)\\
&(x = \rho\cos\theta, y = \rho\sin\theta).
\end{aligned}$$

当 $x = 1,\ y = \tan\theta$ 时

$$\varphi(\tan\theta) = \int_0^1 \int_0^{2\pi} \cos^{-1} \frac{\cos\theta - \mu\cos(\alpha-\theta)}{\sqrt{(1-\mu^2)\sin^2\theta + (\cos\theta - \mu\cos(\alpha-\theta))^2}} dq(\alpha, \mu).$$

问题一变而为怎样的 $\varphi(\tan\theta)$, 我们可以解得变函数 $q(\alpha, \mu)$.

7.10　Poisson 公式

从 Γ 的一般变形函数

$$x' = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad y' = \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}, \tag{1}$$

命

$$x = \cos\theta, \quad y = \sin\theta, \quad x' = \cos\theta', \quad y' = \sin\theta'.$$

如此则得

$$\cos\theta' = \frac{a_1\cos\theta + b_1\sin\theta + c_1}{a_3\cos\theta + b_3\sin\theta + c_3},$$
$$\sin\theta' = \frac{a_2\cos\theta + b_2\sin\theta + c_2}{a_3\cos\theta + b_3\sin\theta + c_3}.$$

所以

$$\tan\theta' = \frac{a_2\cos\theta + b_2\sin\theta + c_2}{a_1\cos\theta + b_1\sin\theta + c_1}. \tag{2}$$

因之, 由 7.1 节的 a, b, c 的关系可得

$$\begin{aligned}\frac{d\theta'}{d\theta} =&((a_1\cos\theta + b_1\sin\theta + c_1)(-a_2\sin\theta + b_2\cos\theta)\\ &- (a_2\cos\theta + b_2\sin\theta + c_2)(-a_1\sin\theta + b_1\cos\theta))\\ &/((a_2\cos\theta + b_2\sin\theta + c_2)^2 + (a_1\cos\theta + b_1\sin\theta + c_1)^2)\\ =&(a_1b_2 - a_2b_1 + (-c_1a_2 + c_2a_1)\sin\theta + (c_1b_2 - c_2b_1)\cos\theta)\\ &/((1 + a_3^2)\cos^2\theta + 2a_3b_3\cos\theta\sin\theta + (1 + b_3^2)\sin^2\theta\\ &+ 2a_3c_3\cos\theta + 2b_3c_3\sin\theta + c_3^2 - 1)\\ =&\frac{1}{a_3\cos\theta + b_3\sin\theta + c_3}.\end{aligned} \tag{3}$$

因此得出

$$1 = \frac{1}{2\pi}\int_0^{2\pi} d\theta' = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|a_3\cos\theta + b_3\sin\theta + c_3|}. \tag{4}$$

假定变换 (1) 把点 (ξ, η) 变为原点, 即

$$a_1\xi + b_1\eta + c_1 = 0, \quad a_2\xi + b_2\eta + c_2 = 0. \tag{5}$$

此点 (ξ, η) 在单位圆内. 由 (5) 可知

$$\xi = -\frac{a_3}{c_3}, \quad \eta = -\frac{b_3}{c_3}.$$

又由

$$a_3^2 + b_3^2 - c_3^2 = -1,$$

可知

$$c_3^2(\xi^2 + \eta^2 - 1) = -1.$$

代入 (3) 式得

$$\frac{d\theta'}{d\theta} = \pm \frac{\sqrt{1 - \xi^2 - \eta^2}}{1 - \xi\cos\theta - \eta\sin\theta}.$$

换 (ξ, η) 为 (x, y), 因此由 (4) 得出：对圆内任一点 (x, y) 常有

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sqrt{1 - x^2 - y^2}}{1 - x\cos\theta - y\sin\theta} d\theta. \tag{6}$$

(由于 $|x\cos\theta - y\sin\theta| < 1$.) 如果有极坐标, 则

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sqrt{1 - \rho^2}}{1 - \rho\cos(\theta - \psi)} d\theta. \tag{7}$$

如果 (x, y) 在圆外, 即当 $\rho > 1$ 时,

$$\int_0^{2\pi} \frac{d\theta}{1 - \rho\cos(\theta - \psi)} = 0. \tag{8}$$

要证明此点十分容易, 因为 $\dfrac{1}{1 - \rho\cos\theta}$ 的不定积分等于

$$\frac{1}{\rho^2 - 1}\log\left|\frac{\sqrt{\rho^2 - 1}\tan\frac{1}{2}\theta + 1 - \rho}{\sqrt{\rho^2 - 1}\tan\frac{1}{2}\theta - 1 + \rho}\right|.$$

因此

$$\begin{aligned}
\int_0^{\pi} \frac{d\theta}{1 - \rho\cos\theta} &= \lim_{\varepsilon\to 0}\left(\int_0^{\arccos\frac{1}{\rho} - \varepsilon} + \int_{\arccos\frac{1}{\rho} + \varepsilon}^{\pi}\right)\frac{d\theta}{1 - \rho\cos\theta} \\
&= \frac{1}{\rho^2 - 1}\lim_{\varepsilon\to 0}\left(\log\left|\frac{\sqrt{\rho^2 - 1}\tan\frac{1}{2}\theta + 1 - \rho}{\sqrt{\rho^2 - 1}\tan\frac{1}{2}\theta - 1 - \rho}\right|\ \Bigg|\begin{matrix}\arccos\frac{1}{\rho} - \varepsilon \\ \\ \arccos\frac{1}{\rho} + \varepsilon\end{matrix}\right) \\
&= \frac{1}{\sqrt{\rho^2 - 1}}\lim_{\varepsilon\to 0}\log\left|\frac{\sqrt{\rho^2 - 1}\tan\frac{1}{2}\left(\arccos\frac{1}{\rho} - \varepsilon\right) + 1 - \rho}{\sqrt{\rho^2 - 1}\tan\frac{1}{2}\arccos\frac{1}{\rho} + \varepsilon + 1 - \rho}\right| \\
&= 0.
\end{aligned}$$

引进一个函数

$$P(\rho,\theta-\psi)=\frac{|\tau|^{\frac{1}{2}}}{1-\rho\cos(\theta-\psi)},\quad \tau=1-\rho^2,$$

称为 Poisson 核. 这函数有以下的一些性质:

(1) 把 $P(\rho,\theta-\psi)$ 看成为极坐标 (ρ,θ) 的函数, 不管圆内圆外, 它适合于 (D).

(2) 在圆周上除去一点 $\theta=\psi$ 外, 处处为 0.

(3) 在整个平面上, 除去一条直线 (特征线之一,$\rho\cos(\theta-\psi)=1$) 外, 处处有限, 但在这特征线上, 它变为无穷.

(4) 我们有关系式

$$\frac{1}{2\pi}\int_0^{2\pi}P(\rho,\theta)d\theta=\begin{cases}0, & 若\ \rho>1,\\ 1, & 若\ \rho<1.\end{cases}$$

如果给了

$$u(\rho,\theta)|_{\rho=1}=\alpha(\theta),$$

作函数

$$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}P(\rho,\theta-\psi)\alpha(\psi)d\psi,\tag{9}$$

在圆内这函数是适合于微分方程 (D) 的.

在圆外, 命

$$\rho=\frac{1}{\cos\eta},\quad 0\leqslant\eta\leqslant\frac{\pi}{2}.$$

则

$$\begin{aligned}P(\eta,\theta-\psi)&=\frac{\sin\eta}{\cos\eta-\cos(\theta-\psi)}\\&=-\frac{\sin\eta}{2\sin\frac{1}{2}(\eta+\theta-\psi)\sin\frac{1}{2}(\eta-\theta+\psi)}\\&=-\frac{1}{2}\left(\tan\frac{1}{2}(\eta+\theta-\psi)+\tan\frac{1}{2}(\eta-\theta+\psi)\right).\end{aligned}$$

由此可得

$$u(\rho,\theta)=F_1(\eta+\theta)+F_2(\eta-\theta),$$

此处

$$F_1(\gamma)=-\frac{1}{4\pi}\int_0^{2\pi}\tan\frac{1}{2}(\gamma-\psi)\alpha(\psi)d\psi,$$

$$F_2(\gamma) = -\frac{1}{4\pi}\int_0^{2\pi} \tan\frac{1}{2}(\gamma+\psi)\alpha(\psi)d\psi.$$

这些奇异积分可能并不存在, 即使存在也可能不能求微商. 但我们可以把公式 (9) 作为在双曲方程 (D) 的广义解来处理.

在 $\alpha(\psi)$ 上加上些条件使 F_1、F_2 存在而且有二阶微商, 这样的 $u(\rho,\theta)$ 就是方程 (D) 的解.

7.11 变型线上给了值的函数

假定给了

$$u(\rho,\theta)|_{\rho=1} = F(\theta) \tag{1}$$

及

$$\lim_{\rho\to 1\pm 0} \frac{u(\rho,\theta)-F(\theta)}{|\tau|^{\frac{1}{2}}} = G(\theta), \tag{2}$$

此处 $F(\theta), G(\theta)$ 在 $\alpha<\theta<\beta$ 间是实解析函数, 由此条件定出适合于 (D) 的函数 $u(\rho,\theta)$.

在圆内命

$$u(\rho,\theta) = U(\lambda,\theta), \tag{3}$$

则

$$U(1,\theta) = F(\theta), \quad \left.\frac{\partial U}{\partial\lambda}\right|_{\lambda=1} = -G(\theta),$$

这就是一个Ковалевская问题, 其解答如下: 先考虑适合于

$$U_1(1,\theta) = F(\theta), \quad \left.\frac{\partial U_1}{\partial\lambda}\right|_{\lambda=1} = 0 \tag{4}$$

的调和函数, 函数

$$U_1(\lambda,\theta) = \sum_{n=0}^{\infty}(-1)^n\frac{(\log\lambda)^{2n}}{(2n)!}F^{(2n)}(\theta) \tag{5}$$

就适合我们的要求, 其理由是

(i) $U_1(1,\theta) = F(\theta)$;

(ii) $\lambda\left.\frac{\partial U_1}{\partial\lambda}\right|_{\lambda=1} = \sum_{n=1}^{\infty}(-1)^n\frac{(\log\lambda)^{2n-1}}{(2n-1)!}\left.F^{(2n)}(\theta)\right|_{\lambda=1} = 0$

及

(iii) $\lambda\frac{\partial}{\partial\lambda}\left(\lambda\frac{\partial U_1}{\partial\lambda}\right) = \sum_{n=1}^{\infty}(-1)^n\frac{(\log\lambda)^{2n-2}}{(2n-2)!}F^{(2n)}(\theta)$

$$= -\frac{\partial^2}{\partial\theta^2}\left(\sum_{m=0}^{\infty}(-1)^m\frac{(\log\lambda)^{2m}}{(2m)!}F^{(2m)}(\theta)\right).$$

由于

$$F(\theta+x)=\sum_{n=0}^{\infty}\frac{F^{(n)}(\theta)}{n!}x^n,$$

可知

$$\begin{aligned}&F(\theta+i\log\lambda)+F(\theta-i\log\lambda)\\ =&\sum_{n=0}^{\infty}\frac{F^{(n)}(\theta)}{n!}(i^n+(-i)^n)(\log\lambda)^n.\end{aligned}$$

由 (5) 得出

$$U_1(\lambda,\theta)=\frac{1}{2}(F(\theta+i\log\lambda)+F(\theta-i\log\lambda)). \tag{6}$$

同法得出适合于

$$U_2(1,\theta)=0,\quad \left.\frac{\partial U_2}{\partial\lambda}\right|_{\lambda=1}=G(\theta)$$

的调和函数是

$$\begin{aligned}U_2(\lambda,\theta)&=\sum_{n=0}^{\infty}(-1)^n\frac{(\log\lambda)^{2n+1}}{(2n+1)!}G^{(2n)}(\theta)\\ &=\frac{1}{2i}(G_1(\theta+i\log\lambda)-G_1(\theta-i\log\lambda)),\end{aligned}$$

此处 $G_1(\theta)=\int_0^\theta G(t)dt$.

因此

$$\begin{aligned}U(\lambda,\theta)=&U_1(\lambda,\theta)-U_2(\lambda,\theta)\\ =&\frac{1}{2}(F(\theta+i\log\lambda)+F(\theta-i\log\lambda))\\ &-\frac{1}{2i}(G_1(\theta+i\log\lambda)-G_1(\theta-i\log\lambda)).\end{aligned}$$

回到原符号, 在圆内

$$\begin{aligned}\sigma(\rho)&=\log\left(\frac{1}{\rho}+\sqrt{\frac{1}{\rho^2}-1}\right)\\ &=-\log\left(\frac{1}{\rho}-\sqrt{\frac{1}{\rho^2}-1}\right)=-\log\lambda.\end{aligned}$$

因此, 当 $\rho\leqslant 1$ 时,

$$u(\rho,\theta)=\frac{1}{2}[F(\theta+i\sigma(\rho))+F(\theta-i\sigma(\rho))]$$

$$+\frac{1}{2i}[G_1(\theta+i\sigma(\rho))-G_1(\theta-i\sigma(\rho))]. \tag{7}$$

在圆外,

$$\begin{aligned}u(\rho,\theta)=&\frac{1}{2}\left[F\left(\theta+\cos^{-1}\frac{1}{\rho}\right)+F\left(\theta-\cos^{-1}\frac{1}{\rho}\right)\right]\\&+\frac{1}{2}\left[G_1\left(\theta+\cos^{-1}\frac{1}{\rho}\right)-G_1\left(\theta-\cos^{-1}\frac{1}{\rho}\right)\right].\end{aligned} \tag{8}$$

极易证明:

$$u(1,\theta)=F(\theta).$$

及

$$\lim_{\rho\to\pm1}\frac{u(\rho,\theta)-u(1,\theta)}{|\tau|^{\frac{1}{2}}}=G(\theta).$$

也就是 (7) 与 (8) 给了本问题的解答.

再看这解答在圆外适用的范围, 由于必须要

$$\alpha\leqslant\theta-\arccos\frac{1}{\rho}<\theta+\arccos\frac{1}{\rho}\leqslant\beta,$$

可知

$$\rho\cos(\alpha-\theta)\geqslant1\geqslant\rho\cos(\beta-\theta).$$

即在单位圆二切线之间, 其一是切于 $\theta=\alpha$, 它一是切于 $\theta=\beta$ 的直线.

假定 $\alpha=0$, 现在看这函数在 $x=1$ 上的情况, 即 $\rho=\dfrac{1}{\cos\theta}$, 则

$$u\left(\frac{1}{\cos\theta},\theta\right)=\frac{1}{2}(F(2\theta)+F(0))+\frac{1}{2}(G_1(2\theta)-G_1(0)),\quad 0<\theta<\beta.$$

7.12 在一特征线上取零值的函数

再考虑适合于

$$u(\rho,\theta)|_{x=1}=0 \tag{1}$$

的函数类.

在圆外. 由于

$$u(\rho,\theta)=F_1\left(\theta_1+\cos^{-1}\frac{1}{\rho}\right)+F_2\left(\theta-\cos^{-1}\frac{1}{\rho}\right),\quad F_2(0)=0.$$

即得

$$u(\rho,\theta)|_{x=1}=F_1(2\theta)=0.$$

因此

$$u(\rho,\theta)=F_2\left(\theta-\cos^{-1}\frac{1}{\rho}\right),\quad F_2(0)=0.$$

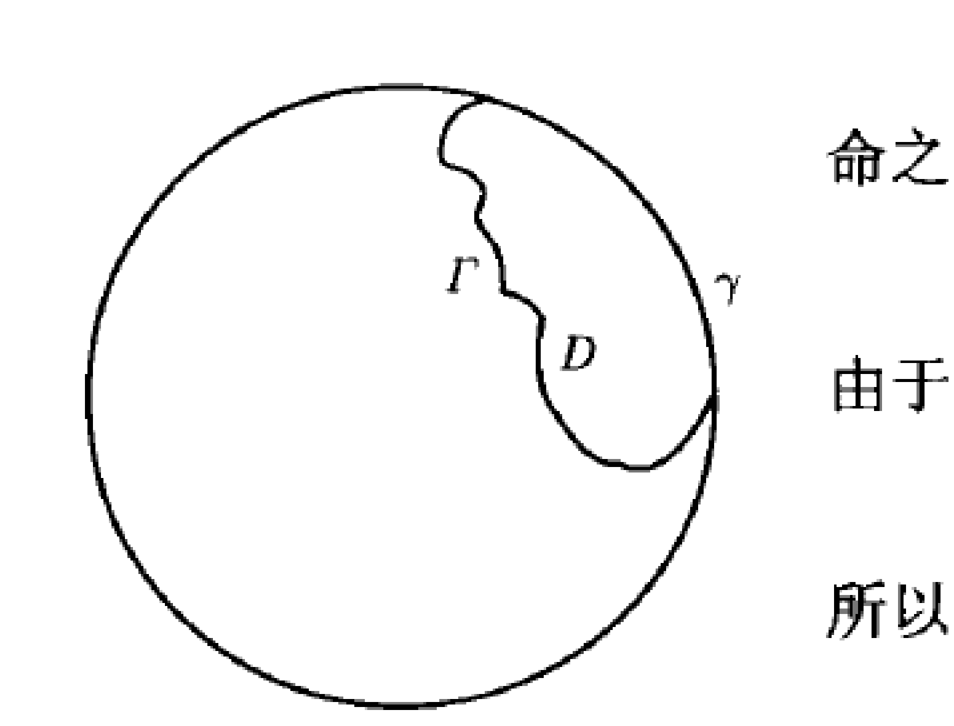
由

$$\lim_{\rho\to 1}\frac{u(\rho,\theta)-u(1,\theta)}{|\tau|^{\frac{1}{2}}}=\lim_{\rho\to 1}\frac{F_2\left(\theta-\cos^{-1}\frac{1}{\rho}\right)}{|\tau|^{\frac{1}{2}}}=F_2'(\theta),$$

命 $u(\rho,\theta)=U(\lambda,\theta)$, 则适合于 (1) 的函数常适合于

$$U(1,\theta)=F_2(\theta),\quad \frac{\partial}{\partial\lambda}U(\lambda,\theta)|_{\lambda=1}=-F_2'(\theta). \tag{2}$$

如果 $u(\rho,\theta)$ 在圆内有一部分存在, 而且包有一段圆弧为边界, 另一部分是一曲线, 在 (λ,θ) 平面上所对应的区域记之为 D, 其边界也有一部分是单位圆的圆弧, 记之为 γ, 其他部分以 $\varGamma$ 记之.



在 D 内 $U(\lambda,\theta)$ 是一个解析函数的实数部分, 而且命之为

$$\varOmega=f(z)=U(\lambda,\theta)+iV(\lambda,\theta).$$

由于

$$\frac{1}{\lambda}\frac{\partial}{\partial\lambda}U(\lambda,\theta)=\frac{\partial}{\partial\theta}V(\lambda,\theta),$$

所以

$$V(1,\theta)=-F_2(\theta)+C,$$

这里 C 是一常数. 不妨假定它等于 0, 保角变换

$$Z=f(z)$$

把 D 变为 D^*, 把 D 的圆弧边界 γ 变为直线

$$U+V=0.$$

假定保角变换是单叶的, 则由 Schwarz 原理解析扩展可以把函数 $f(z)$ 的解析性推到圆外, 假定 $\varGamma'$ 是由 $\varGamma$ 依圆反演出来的曲线, 则在 $\varGamma'$ 与 $\varGamma$ 所围成的区域内 $f(z)$ 定义, 在 $\varGamma$ 上 $f(z)$ 的实数部分已经有了, 在 $\varGamma'$ 上 U 也已定义了, 由 $\varGamma$ 与 $\varGamma'$ 上的值, u 是存在的, 而且是唯一的 (详情与Бицадзе 的工作相仿, 用Келдеыщ-Седов 公式解决这问题).

第 8 讲　形式 Fourier 级数与广义函数

8.1　形式 Fourier 级数

已经不止一次地提到, 给了一个收敛的 Fourier 级数

$$a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),\tag{1}$$

有一个圆内的调和函数

$$a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\rho^n,\quad 0\leqslant\rho<1.\tag{2}$$

但是反过来, 如果 (2) 对任一适合于 $0\leqslant\rho<1$ 的 ρ 收敛, 如果 (2) 收敛, 我们就说 (1) 定义一个广义函数. 因为调和函数是无穷可微的, 因而广义函数也是无穷可微的.

虽然看来简单, 这样定义出来的广义的函数比 Schwarz 的广义函数的范围还大, 更宽广的考虑是从形式 Fourier 级数出发.

一个形式 Fourier 级数

$$\sum_{n=-\infty}^{\infty}a_ne^{in\theta}$$

就定义为一个广义函数, 我们不管它是不是收敛, 是不是在某种意义下可求和. 这一广义函数用 $u(\theta)$ 来表示.

今后常用 $\sum\limits_n$ 表示 $\sum\limits_{n=-\infty}^{\infty}$, 而 $\sum\limits_n{}'$ 表示 $\sum\limits_n$ 中除去一项 $n=0$. 命

$$v(\theta)=\sum_{n=-\infty}^{\infty}b_ne^{in\theta}$$

对任二复数 λ,μ, 形式 Fourier 级数

$$\lambda u(\theta)+\mu v(\theta)=\sum_{n=-\infty}^{\infty}(\lambda a_n+\mu b_n)e^{in\theta}$$

仍然是一广义函数. 所以广义函数是一线性集合.

二广义函数的乘积一般不定义的, 其原因在于

$$\sum_{l+m=n} a_l b_m$$

可能并不收敛.

但如果

$$\sum_n a_n \bar{b}_n$$

收敛 (或在某种意义可求和), 则此值称为二广义函数 $u(\theta)$ 与 $\overline{v(\theta)}$ 的无向积或内积, 以 $(u(\theta), \overline{v(\theta)})$ 表之.

显然有

$$(u(\theta), \overline{v(\theta)}) = \overline{(v(\theta), \overline{u(\theta)})},$$
$$(\lambda u_1(\theta) + \mu u_2(\theta), \overline{v(\theta)}) = \lambda(u_1(\theta), \overline{v(\theta)}) + \mu(u_2(\theta), \overline{v(\theta)}),$$

而且有

$$(u(\theta), \overline{e^{in\theta}}) = a_n.$$

定义

$$(u(\theta), \overline{v(\theta - \psi)}) = (u(\theta + \psi), \overline{v(\theta)}) = \sum_n a_n \bar{b}_n e^{in\psi}$$

为二函数 $u(\theta), v(\theta)$ 的卷积.

最有趣的例子是 Dirac 函数

$$\delta(\theta) = \sum_n e^{in\theta},$$

它使

$$(u(\theta), \overline{\delta(\theta - \psi)}) = \sum_n a_n e^{in\psi} = u(\psi).$$

有时我们也把 $\delta(\theta - \psi)$ 记成为 $\delta_\psi(\theta)$.

广义函数 $u(\theta)$ 的微商的定义是

$$i \sum_n n a_n e^{in\theta},$$

以 $u'(\theta)$ 记之, 显然有

$$(u'(\theta), \overline{v(\theta)}) = i \sum_n n a_n \bar{b}_n = -\sum a_n \overline{(inb_n)} = -(u(\theta), \overline{v'(\theta)}).$$

故立得

$$(u(\theta),\overline{\delta'(\theta-\psi)})=-\overline{(u'(\theta),\delta(\theta-\psi))}=-u'(\psi),$$

及

$$(u(\theta),\overline{\delta^{(v)}(\theta-\psi)})=(-1)^{v}u^{(v)}(\psi).$$

但这样定义的广义函数太广泛了, 不能推出很多有用的结论. 我们现在先引进两类特殊的广义函数.

如果系数 a_n 适合于

$$\max\left(\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}},\overline{\lim_{n\to\infty}}|a_{-n}|^{\frac{1}{n}}\right)\leqslant 1,$$

则所对应的广义函数称为 H 类的广义函数. 或简称 H 型广义函数.

显然, H 类广义函数成一线性集合, 而且 H 类广义函数的微商仍然是 H 类广义函数. 二 H 型广义函数的卷积仍然是 H 型广义函数.

如果有一整数 p, 使 $a_n=O(|n|^p)$, 则所对应的广义函数称为 S 型的广义函数. 这就是 Schwarz 的广义函数.

显然, 一个 S 型广义函数一定是一个 H 型广义函数, 且 S 型广义函数也成为一线性集合且对微分运算及卷积而自封.

附记 H 型广义函数之间定义了两个运算 "加" 和 "卷积", 把 "卷积" 看为 "乘", 则所有的 H 型广义函数成一环.

更明确些, 定义

$$u(\theta)\pm v(\theta)=\sum_n(a_n\pm b_n)e^{in\theta},$$

$$u(\theta)\circ v(\theta)=(u(\psi),v(\theta-\psi))=\sum_n a_nb_ne^{in\theta},$$

交换、结合、分配等定律都不难直接证明. 这环还有单位元

$$\delta(\theta)=\sum_n e^{in\theta}.$$

而

$$\lambda u(\theta)=(u(\theta),\lambda\delta(\psi-\theta)).$$

$e^{in\theta}(n=0,\pm1,\pm2,\cdots)$ 是幂等元, 即 $e^{in\theta}\circ e^{in\theta}=e^{in\theta}$. 这些幂等元是相互正交的, 即 $e^{im\theta}\circ e^{in\theta}=0$(若 $m\neq n$). 这些幂等元的和是单位元 $\delta(\theta)$.

S 型函数也有些性质.

8.2 对 偶 性

两类广义函数 T 与 $\mathring{T}$ 如适合以下的三条件则称为互相对偶：(i) 对诸 $u \in T$ 及诸 $v \in \mathring{T}$, $(u, \bar{v})$ 常收敛. (ii) 若 $(u, \bar{v})$ 对所有的 $u \in T$ 常收敛, 则可以推出 $v \in \mathring{T}$ 及 (iii) 若 $(u, \bar{v})$ 对所有的 $v \in \mathring{T}$ 常收敛, 则 $u \in T$.

例 1 所有广义函数成一类 K 和所有的由有限 Fourier 级数所成的类 $\mathring{K}$ 之间是有对偶关系的.

例 2 命 $p > 1$ 及 $p' = p/p-1$, 则 L^p 与 $L^{p'}$ 之间是有对偶关系的. 这些 L^p 表示适合于

$$\sum_n |a_n|^p < \infty$$

的 $u(\theta)$ 的函数类.

例 3 如果把 $\sum\limits_n a_n \bar{b}_n$ 的收敛性改为 $(c, 1)$ 求和法, 则还有以下的一些对偶类: (a) B 与 L(此处 B 表示囿函数的 Fourier 级数所成的类), (b) C 与 St 成一对偶类 (此处 C 表示连续函数的 Fourier 级数所成的类, S_t 表示 Fourier-Stieltjes 级数所成的类).

总起来, 具有以下的关系

$$C \subset L^\infty = B \subset L^{p'} \subset L^2 \subset L^p \subset L \subset St.$$

容易证明, 一类与其对偶数重合必然是 L^2.

定理 1 命 $\varphi(n)$ 表一递增正函数, 当 n 趋向无穷时, $\varphi(n)$ 趋向无穷. 并且假定对任一 $\delta > 0$, 级数

$$\sum_{n=1}^{\infty} \frac{1}{(\varphi(n))^\delta}$$

常收敛. 命 T 代表适合于

$$\log |a_n| = o(\log \varphi(|n|))$$

的广义函数所成的类, 而 $\mathring{T}$ 是适合于

$$\log \varphi(|n|) = O\left(\log \frac{1}{|b_n|}\right)$$

的广义函数所成的类, 则 T 与 $\mathring{T}$ 之间有对偶关系.

证 (i) 由定义, 对任一 $\varepsilon > 0$, 当 n 充分大时常有

$$|a_n| \leqslant (\varphi(|n|))^\varepsilon,$$

又有一数 $c>0$ 使

$$|b_n| \leqslant \frac{1}{(\varphi(|n|))^c}.$$

所以 $\sum\limits_n a_n\bar{b}_n$ 是收敛的.

(ii) 假定 v 不属于 $\mathring{T}$, 则有一数列 n_v, 使

$$\lim_{v\to\infty} \frac{\log\varphi(|n_v|)}{\log\dfrac{1}{|b_{n_v}|}} = \infty.$$

取 $a_{n_v} = 1/b_{n_v}$ 及其他 $a_n = 0$, 如此所定义的广义函数 $u(\theta)$ 属于 T 而 $\sum\limits_n a_n\bar{b}_n$ 发散.

(iii) 假定 u 不属于 T, 则有一数列 n_v, 使

$$\log|a_{n_v}| \geqslant c\log\varphi(|n_v|), \quad c>0.$$

取 $b_{n_v} = \dfrac{1}{a_{n_v}}$ 及其他 $b_n = 0$, 如此所定义的广义函数 $v(\theta)$ 属于 $\mathring{T}$ 而 $\sum\limits_n a_n\bar{b}_n$ 发散.

在定理 1 中取 $\varphi(n) = e^n$, 则类 T 立刻变为类 H. 盖由

$$\log|a_n| = o(|n|),$$

可知

$$\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}} \leqslant 1.$$

又关系

$$|n| = O\left(\log\frac{1}{|b_n|}\right),$$

与次之关系等价

$$\overline{\lim_{|n|\to\infty}}|b_n|^{\frac{1}{|n|}} < 1.$$

因此得

定理 2 H 类的对偶类 $\mathring{H}$ 是适合于以下条件的广义函数所组成:

$$\max\left(\overline{\lim_{n\to\infty}}|b_n|^{\frac{1}{n}}, \overline{\lim_{n\to\infty}}|b_{-n}|^{\frac{1}{n}}\right) < 1.$$

取 $\varphi(n) = e^{n^p} (p>1)$ 所得的类 T 以 G_p 表之, 由定理 1 可推出:

定理 3 G_p 是由适合于

$$\overline{\lim_{|n|\to\infty}}|a_n|^{|n|^{-p}} \leqslant 1$$

的广义函数所成的类, 它的对偶类 $\overset{\circ}{G}_p$ 是由适合于

$$\overline{\lim_{|n|\to\infty}}|b_n|^{|n|^{-p}} < 1$$

的广义函数所成的.

类 G_p 是由 Гелфанд-Шилов 所引进的.

与定理 1 的证法相仿可得

定理 4 命 $\psi(n)$ 表一递增正函数当 n 趋无穷, 并且假定有一正数 λ 使

$$\sum_{n=1}^{\infty}\frac{1}{(\psi(n))^\lambda}$$

收敛. 命 T 表适合以下条件的广义函数类

$$\log|a_n| = O\left(\log\psi(|n|)\right),$$

及 $\overset{\circ}{T}$ 为适合于

$$\log\psi(|n|) = o\left(\log\frac{1}{|b_n|}\right)$$

的广义函数所成的类, 则类 T 与类 $\overset{\circ}{T}$ 是有对偶关系的.

在定理 4 中取 $\psi(n) = n$, 则类 T 就是类 S, 故得

定理 5 S 类的对偶类 $\overset{\circ}{S}$ 是由适合以下条件的广义函数所组成的, 对任一 $q > 0$ 常有

$$b_n = O\left(\frac{1}{|n|^q}\right).$$

再在定理 4 中取 $\psi(n) = e^n$, 所得出的类 T 以 I 表之, 由定理 4 可得

定理 6 I 类的对偶类 $\overset{\circ}{I}$ 是由适合于

$$\lim_{n\to\infty}|b_n|^{-|n|^{-1}} = \infty$$

的函数所组成的.

附记 1 类还可以分得更细致, 例如, 在定理 4 中把条件改为

$$\overline{\lim_{n\to\infty}}\frac{\log|a_n|}{\log\psi(|n|)} \leqslant \rho,$$

并无任何困难可以找出这类的对偶类.

附记 2 极易证明, $\overset{\circ}{H}$ 类 (或 $\overset{\circ}{S}$ 类) 也是线性集合而且对微分运算及卷积而自封的, 它虽然成一环, 但并不包有单位元. $\overset{\circ}{H}$ 是 H 的理想子, 命 H^* 表示 H 中的任一理想子. 如果 H^* 中所包有的广义函数都是普通函数, 也就是这些广义函数的形式 Fourier 级数都是收敛的, 则 H^* 称为函数类的理想子, 易见: $\overset{\circ}{H}$ 是 H 的最大的函数类理想子.

8.3 H 型广义函数的意义

对应一个 H 型广义函数 $u(\theta)$, 我们有一个函数

$$u(r,\theta)=\sum_n a_n e^{in\theta}r^{|n|},\quad 0\leqslant r<1,\quad 0\leqslant\theta<2\pi,$$

此乃单位圆的调和函数.

故一个 H 型广义函数可以视为一个圆内调和函数的边界值函数.

同法一个 $\mathring{H}$ 型广义函数 $v(\theta)$(它是一个普通意义的函数) 对应于一个在较大同心圆中的调和函数.

又显然, 对应于一个 $\mathring{I}$ 型广义函数 $v(\theta)$, 我们有一个处处调和的函数

$$\sum_n b_n e^{in\theta}r^{|n|}.$$

这种函数也称为调和整函数. 所以 $\mathring{I}$ 型函数是调和整函数在单位圆周上的数值.

广义函数

$$\delta_\psi(\theta)$$

属于 H, 但不属于 $\mathring{H}$, 这广义函数所对应的调和函数就是

$$\sum_n e^{in(\theta-\psi)}r^{|n|}=\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}.$$

这就是习知的 Poisson 核, 以 $P(r,\theta)$ 表之.

命 $f(\theta)$ 表一连续 (或可积) 函数, 其 Fourier 系数为 b_n, 则对任一 $u(\theta)\in H$ 常有

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{f(\theta)}d\theta&=\frac{1}{2\pi}\int_0^{2\pi}\sum_n a_n e^{in\theta}r^{|n|}\overline{f(\theta)}d\theta\\&=\sum_n a_n r^{|n|}\frac{1}{2\pi}\int_0^{2\pi}\overline{f(\theta)}e^{in\theta}d\theta\\&=\sum_n a_n\bar{b}_n r^{|n|}.\end{aligned}$$

如果 $(u(\theta),\overline{f(\theta)})$ 收敛, 则由 Abel 定理可知

$$(u(\theta),\overline{f(\theta)})=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{f(\theta)}d\theta.$$

更一般些, 对 H 中任二广义函数 $u(\theta)$ 及 $v(\theta)$, 对 $r<1$, $r'<1$ 常有

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(r',\theta)}d\theta=\sum_n a_n\bar{b}_n(rr')^{|n|}.$$

如果 $(u,\bar{v})$ 存在, 则

$$(u,\bar{v})=\lim_{r\to1}\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(r,\theta)}d\theta.$$

又对 $u\in H$ 及 $v\in \overset{\circ}{H}$, 则有一 $\delta>0$ 使 $0\leqslant r'<1+\delta$ 时, $v(r',\theta)$ 调和, 取 $r=\dfrac{1}{1+\frac{1}{2}\delta}$ 及 $r'=1+\frac{1}{2}\delta$, 可知

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(r',\theta)}d\theta=\sum_n a_n\bar{b}_n=(u(\theta),\overline{v(\theta)}).$$

8.4　S 型广义函数的意义

任一连续函数 $u(\theta)$ 的 Fourier 系数 a_n 适合于

$$|a_n|\leqslant\frac{1}{2\pi}\int_0^{2\pi}|f(\theta)|d\theta=O(1).$$

作为一个 S 型广义函数, 其 p 阶微商的 Fourier 系数 $a_n^{(p)}$ 适合于

$$a_n^{(p)}=O(|n|^p).$$

反之, 如果

$$a_n=O(|n|^p),$$

则 $u(\theta)-a_0$ 是广义函数

$$\sum_n{}'\frac{a_n}{(in)^{p+2}}e^{in\theta}$$

的 $p+2$ 阶微商. 而这级数一致收敛, 收敛于一个连续函数. 因此, 所谓 S 型的广义函数类实质上就是连续函数所定义的广义函数的有限次微商的集合.

同法可以证明: $\overset{\circ}{S}$ 型的广义函数类就是无穷次可微的函数所成的集合.

由上节的结果已知: 若 $u\in S$, $v\in\overset{\circ}{S}$, 则

$$(u,\bar{v})=\lim_{r\to1}\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(\theta)}d\theta.$$

如果 $u(\theta)-a_0$ 是连续函数 $w(\theta)$ 的 p 次微商, 则由分部积分可知

$$\begin{aligned}(u,\bar{v}) &= a_0\bar{b}_0+\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}(u(r,\theta)-a_0)\overline{v(\theta)}d\theta\\ &= a_0\bar{b}_0+\lim_{r\to 1}\frac{(-1)^p}{2\pi}\int_0^{2\pi}w(r,\theta)\overline{v^{(p)}(\theta)}d\theta.\\ &= a_0\bar{b}_0+\frac{(-1)^p}{2\pi}\int_0^{2\pi}w(\theta)\overline{v^{(p)}(\theta)}d\theta.\end{aligned}$$

就是我们可以用普通的运算表达出来.

8.5 致 零 集

定义 单位圆周上的一个开区间 $a<\theta<b$ 称为一个 H 型广义函数的致零区间, 如果在 $a<\theta<b$ 中的任一闭区间中一致地

$$\lim_{r\to 1}u(r,\theta)=0.$$

一点 θ_0 称为 $u(\theta)$ 的支点, 如果没有包有 θ_0 的致零区间存在.

所有的 $u(\theta)$ 的致零区间的总集合称为函数 $u(\theta)$ 的致零集. 这是一个开集, 其补集称为支点集. 显然支点集的任一点都是支点.

例 $\delta_\psi(\theta)$ 是一个以 $\theta=\psi$ 为唯一支点的广义函数. 其理由是, 当 $\theta\neq\psi$ 时, 当 $r\to 1$ 时

$$P(r,\theta)=\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}$$

趋于 0.

定理 1 H 型的仅有一个支点 $\theta=\psi$ 的广义函数可以表成为

$$u(\theta)=\sum_{v=0}^{\infty}C_v\overline{D^{(v)}[\delta_\psi(\theta)]},$$

此处 $D^{(v)}[\delta_\psi(\theta)]$ 是 $\delta^0_\psi(\theta)=\delta_\psi(\theta),\delta'_\psi(\theta),\cdots,\delta^{(v)}_\psi(\theta)$ 的线性组合, 而 $\delta^{(v)}_\psi(\theta)$ 是 $\delta_\psi(\theta)$ 的 v 阶微商, 更确切些, 对任一 $v\in\overset{\circ}{H}$, 常有

$$(u(\theta),\overline{v(\theta)})=\sum_{v=0}^{\infty}C_v\overline{D^{(v)}(v(\psi))},$$

这里

$$D^{(v)}(v(\psi))=\frac{1}{v!}\left[\frac{d^v v(\psi+\sin^{-1}x)}{dx^v}\right]_{x=0},$$

以上的级数是收敛的.

定理 2　S 型的仅有一个支点 $\theta=\psi$ 的广义函数可以表成为

$$u(\theta)=\sum_{v=0}^{l}c_v\overline{\delta^{(v)}(\theta)},$$

这里 l 是一非负整数. 更确切些, 对任一 $v\in\overset{\circ}{S}$, 则有

$$(u(\theta),\overline{v(\theta)})=\sum_{v=0}^{l}c_v\overline{v^{(v)}(\theta)}$$

二定理的证明:

并不失其普遍性, 我们可以假定 $\psi=0$. 我们把区间移成为 $-\pi,\pi$.

对任一已与的 $\varepsilon>0$, 当 $r\to 1$ 时函数在 $|\theta|>\varepsilon$ 的部分上一致趋近于 0, 故

$$\lim_{r\to 1}\int_{-\pi}^{\pi}u(r,\theta)\overline{v(\theta)}d\theta=\lim_{r\to 1}\int_{-\varepsilon}^{\varepsilon}u(r,\theta)\overline{v(\theta)}d\theta,$$

当 ε 充分小时, 在 $|\theta|\leqslant\varepsilon$ 中, 我们把函数 $v(\theta)$ 展开成

$$v(\theta)=\sum_{v=0}^{l}D^{(v)}(v(0))\sin^v\theta+R(\theta),$$

这里 $R(\theta)=O(\varepsilon^{l+1})$,

$$D^{(v)}(v(0))=\frac{1}{v!}\left(\frac{d^v v(\sin^{-1}x)}{dx^v}\right)\bigg|_{x=0}$$

在定理 1 的假定下, 这级数可以展到无穷, 而且当 $|\theta|\leqslant\varepsilon$ 时一致收敛.

由于

$$\begin{aligned}C_v&=\lim_{r\to 1}\int_{-\varepsilon}^{\varepsilon}(\sin\theta)^v u(r,\theta)d\theta=\lim_{r\to 1}\int_{-\pi}^{\pi}(\sin\theta)^v u(r,\theta)d\theta\\&=\lim_{r\to 1}\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^v u(r,\theta)d\theta\\&=\lim_{r\to 1}\frac{1}{(2i)^v}\sum_{t=0}^{v}\left(\begin{array}{c}v\\t\end{array}\right)\int_{-\pi}^{\pi}e^{it\theta}e^{-i(v-t)\theta}u(r,\theta)d\theta\\&=\frac{1}{(2i)^v}\sum_{t=0}^{v}\left(\begin{array}{c}v\\t\end{array}\right)a_{v-2t}.\end{aligned}$$

故得定理 1 (并且我们有了 C_v 的表达式).

在证明定理 2 时, 我们命

$$R^*(\theta)=\begin{cases}R(\theta), & 当\ |\theta|<\varepsilon,\\ 0, & 其他点.\end{cases}$$

则

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon}R(\theta)u(r,\theta)d\theta&=\int_{-\pi}^{\pi}R^*(\theta)u(r,\theta)d\theta\\&=\int_{-\pi}^{\pi}R^*(\theta)(u(r,\theta)-a_0)d\theta+a_0\int_{-\varepsilon}^{\varepsilon}R(\theta)d\theta.\end{aligned}$$

最后一项显然随 ε 而趋于 0, 又命 $w(\theta)$ 表一连续函数其 l 次微商等于 $u(r,\theta)-a_0$, 如此则

$$\begin{aligned}\lim_{r\to1}\left|\int_{-\pi}^{\pi}R^*(\theta)(u(r,\theta)-a_0)d\theta\right|&=\lim_{r\to1}\left|\int_{-\pi}^{\pi}R^{*(l)}(\theta)w(r,\theta)d\theta\right|\\&\leqslant\int_{-\pi}^{\pi}|R^{*(l)}(\theta)w(\theta)|d\theta=O(\varepsilon),\end{aligned}$$

故得定理 2.

8.6 其他类型的广义函数

定义 命 $\rho>0$, 如果

$$\overline{\lim_{n\to\infty}}\frac{\log|a_n|}{n\log n}<\frac{1}{\rho},\quad\overline{\lim_{n\to\infty}}\frac{\log|a_{-n}|}{n\log n}<\frac{1}{\rho},$$

则 $u(\theta)$ 称为 J_ρ 类的广义函数, 或 J_ρ 型广义函数.

显然, J_ρ 型广义函数成一线性集合, 其微商仍然是广义函数. 但注意, 两个 J_ρ 型广义函数的卷积不一定是广义函数.

定理 1 J_ρ 的对偶类 $\mathring{J}_\rho$ 的广义函数适合以下条件

$$\underline{\lim_{n\to\infty}}\frac{\log\dfrac{1}{|b_n|}}{n\log n}\geqslant\frac{1}{\rho},\quad\underline{\lim_{n\to\infty}}\frac{\log\dfrac{1}{|b_{-n}|}}{n\log n}\geqslant\frac{1}{\rho}.$$

证 (1) 由 J_ρ 的定义, 对一 $\delta>0$, 存在 $n_0(\delta)$ 使 $n\geqslant n_0(\delta)$ 时,

$$\frac{\log|a_n|}{n\log n}<\frac{1}{\rho}-\delta,$$

即

$$|a_n|<n^{(\frac{1}{\rho}-\delta)n}.$$

另一方面, 由 $\mathring{J}_\rho$ 的定义, 当 n 充分大时,

$$|b_n| < n^{-(\frac{1}{\rho}-\frac{1}{2}\delta)n},$$

因此 $\sum\limits_n a_n b_n$ 是收敛的.

(2) 假定

$$\overline{\lim_{n\to\infty}} \frac{\log|a_n|}{n\log n} \geqslant \frac{1}{\rho},$$

则有一数列 n_v 使

$$\lim_{v\to\infty} \frac{\log|a_{nv}|}{n_v \log n_v} = \frac{1}{\sigma} \geqslant \frac{1}{\rho}.$$

取 $b_{n_v} = 1/a_{n_v}$, 其他的 $b_n = 0$, 则所得出的 $v(\theta) \in \mathring{J}_\rho$, 而且 $\sum a_n b_n$ 发散.

(3) 假定

$$\underline{\lim_{n\to\infty}} \frac{\log \dfrac{1}{|b_n|}}{n\log n} < \frac{1}{\rho},$$

则有一数列 n_v 使

$$\lim_{n\to\infty} \frac{\log \dfrac{1}{|b_{n_v}|}}{n_v \log n_v} = \frac{1}{\tau} < \frac{1}{\rho},$$

取 $a_{n_v} = 1/b_{n_v}$ 及其他 $a_n = 0$. 则定义 $-u(\theta) \in J_\rho$, 而 $\sum a_n b_n$ 发散.

对应于 J_ρ 类中的一个广义函数 $u(\theta)$, 我们引进一个函数

$$u_\rho(r,\theta) = \sum_{m=0}^{\infty} p_m r^m \sum_{|n|\leqslant m} a_n e^{in\theta},$$

这里

$$p_m = 1/(m!)^{\frac{1}{\rho}}.$$

由于有一 $\delta > 0$ 使

$$\left| p_m \sum_{|n|\leqslant m} a_n e^{in\theta} \right| \leqslant \frac{\sum\limits_{|n|\leqslant m} |a_n|}{(m!)^{1/\rho}} = O\left(\frac{m \cdot m^{(\frac{1}{\rho}-\delta)m}}{m^{\frac{1}{\rho}m}} \right) = O(m^{-\frac{1}{2}\delta m}),$$

可知在平面上任一紧致集 $u_\rho(r,\theta)$ 是一致 (绝对) 收敛的级数.

定理 2　对任一 $u \in J_\rho$ 及任一 $v \in \mathring{J}_\rho$, 则

$$(u, \bar{v}) = \lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} u_\rho(r,\theta) v(\theta) d\theta / J_\rho(r),$$

此处

$$J_\rho(r)=\sum_{m=1}^{\infty}\frac{r^m}{(m!)^{1/\rho}}.$$

证　易知

$$\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{v(\theta)}d\theta=\sum_l\sum_{n=0}^{\infty}p_mr^m\sum_{|n|\leqslant m}a_n\bar{b}_l\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-l)\theta}d\theta$$
$$=\sum_{m=0}^{\infty}p_mr^m\sum_{|n|\leqslant m}a_n\bar{b}_n.$$

由广义 Borel 求和定理 (见本讲附录定理 3), 可以推出本定理.

同样我们可以定义致零集, 即为在开区间 $a<\theta<b$ 中的任一子闭区间中, 当 $r\to\infty$ 时,

$$u_\rho(r,\theta)/J_\rho(r)$$

一致趋于 0. 这样开区间的总集合称为广义函数 $u(\theta)$ 的致零集, 而致零集的补集中的点称为支点.

附记　由整函数论的知识可知 $\mathring{J}_\rho$ 就是所有的阶 $\leqslant\rho$ 的调和整函数的集合, 而 J_ρ 中所施行的求和法就是广义 Borel 求和法.

前已提过两个 J_ρ 广义函数的卷积不一定是 J_ρ 广义函数, 但如果考虑适合于

$$\log|a_n|=O(|n|\log|n|)$$

的广义函数所成的集 J, 则 J 广义函数有以下的三性质: 线性集; 对微分运算自封; 卷积仍然是 J 广义函数, 而 $\mathring{J}$ 则由所有的零阶整函数所组成的.

8.7　继　　续

我们还可以推得更广泛些. 命

$$Q(r)=\sum_{n=0}^{\infty}q_nr^n,\quad q_n\geqslant 0$$

表一幂级数, 对所有的 r 都收敛.

以 I_Q 表适合于以下条件的广义函数的集合

$$\varlimsup_{n\to\infty}\frac{\log(|a_n|q_n)}{n\log n}<0,\quad \varlimsup_{n\to\infty}\frac{\log(|a_{-n}|q_n)}{n\log n}<0.$$

同法可证:

定理 1　类 I_Q 的对偶类 $\mathring{I}_Q$ 是由适合于

$$\underline{\lim}_{n\to\infty} \frac{\log\left(\dfrac{q_n}{|b_n|}\right)}{n\log n} \geqslant 0, \quad \underline{\lim}_{n\to\infty} \frac{\log\left(\dfrac{q_n}{|b_{-n}|}\right)}{n\log n} \geqslant 0$$

的广义函数所组成.

又易证, 对 $u \in I_Q, v \in \mathring{I}_Q$ 常有

$$(u, \bar{v}) = \lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} u_Q(r,\theta)\overline{v(\theta)}d\theta/Q(r),$$

这里

$$u_Q(r,\theta) = \sum_{m=0}^{\infty} q_m r^m \sum_{|n|\leqslant m} a_n e^{in\theta}.$$

随便你给了怎样的广义函数, 总可以选得合适的 q_n 使

$$\overline{\lim_{n\to\infty}} \frac{\log(|a_n|q_n)}{n\log n} < 0, \quad \overline{\lim_{n\to\infty}} \frac{\log(|a_{-n}|q_n)}{n\log n} < 0.$$

换一句话说, 任何复杂的广义函数, 我们都有办法处理的.

8.8　极　　限

在一个广义函数类 T 中定义极限的方法如下: 首先我们假定 T 尽由普通函数所组成的.

$u_v(\theta)(v = 1, 2, \cdots)$ 是 T 中的一个函数贯. $u(\theta) \in T$ 称为这贯的极限的定义是: 对对偶类 $\mathring{T}$ 中任一函数 $v(\theta)$ 恒有

$$\lim_{v\to\infty} (u_v, \bar{v}) = (u, \bar{v}).$$

至于 T 的对偶类 $\mathring{T}$ 中的广义函数, 通常是普通意义下的函数, 我们在不同的情况下, 分别给以普通函数的收敛作为其极限.

(i) 假定 $v_v(r,\theta)(v = 1, 2, \cdots)$ 是一个函数贯, 在一个包有闭单位圆的域内都是调和的, 且在这域内的任一紧致子集上一致收敛, 则其极限函数 $v(r,\theta)$ 显然在这域中是调和的. 我们称 $v(\theta)$ 为 $v_v(\theta)$ 的极限, 以 $v_v(\theta) \to v(\theta)(\mathring{H})$ 表之.

对任一 $u(\theta) \in H$, 我们取 $\delta(>0)$ 充分小, 使所有的 $v_v(r,\theta)$ 定义的域包含有以 $r' = 1 + \dfrac{1}{2}\delta$ 为半径的闭圆. 由 8.3 节可知

$$(u(\theta), v_v(\theta)) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta)\overline{v_v(r',\theta)}d\theta, \quad r = \frac{1}{1 + \dfrac{1}{2}\delta}.$$

由假设 $v_v(r',\theta)$ 在半径为 r' 的闭圆内一致收敛, 故有

$$\lim_{v\to\infty}(u,\bar{v}_v)=\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{v(r',\theta)}d\theta=(u,\bar{v}).$$

(ii) 假定 $v_v(r,\theta)(v=1,2,\cdots)$ 是一个函数贯, 在单位圆内调和, 在圆周上有无穷次微商, 并且对任一 $p\geqslant 0$,

$$\frac{\partial^p}{\partial\theta^p}v_v(r,\theta)$$

在闭单位圆上一致收敛, 则不难推出其极限函数 $v(r,\theta)$ 也在单位圆内调和, 在圆周上有无穷次微商. 我们定义为 $v_v(\theta)\to v(\theta)(\mathring{S})$.

任与 S 类的一个广义函数

$$u(\theta)=\sum_n a_n e^{in\theta},$$

并设

$$v_v(\theta)=\sum_n b_n^{(v)}e^{in\theta},\quad v(\theta)=\sum_n b_n e^{in\theta}.$$

由于 $v_v(\theta)$ 在圆周上一致收敛, 所以

$$\lim_{v\to\infty}b_0^{(v)}=\lim_{v\to\infty}\frac{1}{2\pi}\int_0^{2\pi}v_v(\theta)d\theta=\frac{1}{2\pi}\int_0^{2\pi}v(\theta)d\theta=b_0.$$

由 8.4 节知, 不妨假定 $u(\theta)$ 是连续函数 $w(\theta)$ 的 p 次微商, 我们有

$$(u,\bar{v}_v)=a_0b_0^{(v)}+\frac{(-1)^p}{2\pi}\int_0^{2\pi}w(\theta)\overline{v_v^{(p)}(\theta)}d\theta.$$

我们假设了: 在圆周上 $v_v^{(p)}(\theta)$ 也一致收敛于 $v^{(p)}(\theta)$, 所以

$$\lim_{v\to\infty}(u,\bar{v}_v)=a_0b_0+\frac{(-1)^p}{2\pi}\int_0^{2\pi}w(\theta)\lim_{v\to\infty}\overline{v_v^{(p)}(\theta)}d\theta=(u,\bar{v}).$$

这就是说, 如果

$$v_v(\theta)\to v(\theta)(\mathring{S}),$$

则恒有①

$$\lim_{v\to\infty}(u,\bar{v}_v)=(u,\bar{v}).$$

---

① 不难证明, 这里所定义的 $\mathring{S}$ 类的极限是与 Schwarz 所定义的极限是一样的. 但值得注意的, 这里所定义的 Schwarz 广义函数类 S 时, 并未假定这泛函的连续性, 而这里证明了连续性是必然的, 不必在定义中假定的.

(iii) 假定 $v_v(r,\theta)(v=1,2,\cdots)$ 是一个整个平面上调和的函数贯. 在任一紧致集上, 这贯收敛于一个函数 $v(r,\theta)$, 这样我们定义

$$v_v(\theta)\to v(\theta)\quad(\overset{\circ}{I}),$$

从 (i) 的证明中知道必然有

$$\lim_{v\to\infty}(u,\bar{v}_v)=(u,\bar{v}).$$

(iv) 假定 $v_v(r,\theta), v=1,2,\cdots$ 是一个零阶整调和函数贯, 在平面任一紧致集上一致收敛于一函数 $v(r,\theta)$, 假定它也是零阶的, 则定义 $v_v(\theta)\to v(\theta)(\overset{\circ}{J})$, 同样有

$$\lim_{v\to\infty}(v,\bar{v}_v)=(u,\bar{v}).$$

由以上的这些例子, 可以看出对应于任何一广义函数 T 的对偶类 $\overset{\circ}{T}$, 我们总可以用相仿的方法适当地引进 $\overset{\circ}{T}$ 中的极限概念, 使得下面的关系成立

$$\lim_{v\to\infty}(u,\bar{v}_v)=(u,v),$$

这里不一一列举, 读者可试作之.

8.9　附　　记

(1) 从复变函数论的眼光来看, 我们之所以引进这些广义函数类是十分自然的. 因为

类	类中函数的说明
$\overset{\circ}{K}$	有限和 (调和多项式)
$\overset{\circ}{G}_p$	零阶 p 型的调和整函数
$\overset{\circ}{J}$	零阶调和整函数
$\overset{\circ}{I}$	所有的调和整函数
$\overset{\circ}{H}$	调和函数的有则域是以闭单位圆的点为其内点者

并且有关系

$$\overset{\circ}{K}\subset\overset{\circ}{J}\subset\overset{\circ}{I}\subset\overset{\circ}{H}\subset H\subset I\subset J\subset K.$$

利用整函数的阶在 $\overset{\circ}{J}$ 与 $\overset{\circ}{I}$ 之间还可以插进 J_ρ.

在 $\overset{\circ}{H}$ 与 H 之间还可以利用调和函数的边界函数的性质插入其他类, 如 8.2 节例 3 所列.

另一方面从发散级数的求和理论, 我们的讨论也有它的系统性的意义.

(2) 由 8.2 节的定理 1 与 4 可知, 从系数的无穷大的阶来分类也是极自然的、有系统性的, 主要的类是

类	$\log\|a_n\|$ 的阶
S	$O(\log n)$
H	$o(n)$
I	$O(n)$
J	$O(n\log n)$
G_p	$o(n^p), p>1$

(3) 由保角变换的基本定理, 我们所讨论的类 H, 实质上并不限于单位圆. 我们可以考虑任何平面上的有一点以上边界的单连通域, 特别是上半平面与实数轴.

另一方面, 由于 Fourier 级数与 Fourier 积分的相似性质, 我们可以直接考虑"形式 Fourier 积分":

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a(t)e^{itx}dt.$$

如果 $\log|a(t)|=o(t)$, 则 $u(x)$ 称为属于 H 类, 如果 $\log|a(t)|=O(\log|t|)$, 则 $u(x)$ 称为属于 S 类, 其对应的调和函数是

$$u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a(t)e^{itx-|t|y}dt,\quad y>0.$$

(4) 更一般些, 从推广 8.2 节的定理 1 与 4 入手: 命 R_n 表 n 维实 Euclid 空间, $t=(t_1,\cdots,t_n)$ 表其中的一点, 又命

$$\tau=\sqrt{t_1^2+\cdots+t_n^2}.$$

与定理 1 相仿我们有以下的结果.

命 $\varphi(\tau)$ 表一单变数 $\tau(\geqslant 0)$ 的正递增函数, 且对任一 $\delta>0$ 积分

$$\int_0^{\infty}(\varphi(\tau))^{-\delta}\tau^{n-1}d\tau \tag{1}$$

常收敛.

命 A 代表适合以下条件的函数 $a(t)$: (i) 在任一有限区间中 $a(t)$ 是平方可积及 (ii) 除一测度为零的集合外, 当 τ 充分大时

$$\log|a(t)|=o(\log\varphi(\tau)). \tag{2}$$

又命 B 代表适合以下条件的函数 $b(t)$: (i) 在任一有限区间中 $b(t)$ 是平方可积及 (ii) 除一测度为零的集合外, 当 τ 充分大时

$$\log\varphi(\tau)=O\left(\log\frac{1}{|b(t)|}\right). \tag{3}$$

如此, 则 A 与 B 之间有以下的三性质: (i) 如果 $a(t)\in A, b(t)\in B$, 则

$$\int_{-\infty}^{\infty}\cdots\int a(t)\overline{b(t)}dt_1\cdots dt_n<\infty, \tag{4}$$

(ii) 如果对 B 中任一 $b(t)$, (4) 式常收敛, 则 $a(t)\in A$, (iii) 如果对 A 中任一 $a(t)$, (4) 式常收敛, 则 $b(t)\in B$.

证明从略, 还有与定理 4 相仿的结果.

命 $\overset{\circ}{K}$ 表

$$v(x)=\frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty}\cdots\int b(t)e^{-itx'}dt_1\cdots dt_n$$

所成的广义函数, 此处 $tx'=t_1x_1+\cdots+t_nx_n$. 这积分是绝对收敛的.

积分

$$\int_{-\infty}^{\infty}\cdots\int a(t)\overline{b(t)}dt$$

可以看成为 $a(t)(\in A)$ 的一个线性泛函, 因而定义了一个广义函数 $u(x)$, 且

$$(u(x),\overline{v(x)})=\int_{-\infty}^{\infty}\cdots\int a(t)\overline{b(t)}dt_1\cdots dt_n,$$

这 $u(x)$ 可以用形式 Fourier 积分

$$u(x)=\frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty}\cdots\int a(t)e^{-itx'}dt_1\cdots dt_n$$

来表达. 这样的 $u(x)$ 可以用以下的方法实现出来:

(1) 若对任一 $\varepsilon>0, a(t)=O(e^{\varepsilon|t|})$, 则用

$$u(x,y)=\frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty}\cdots\int a(t)e^{-itx'-|t|y'}dt_1\cdots dt_n,$$

这里 $|t| = (|t_1|, \cdots, |t_n|)$, 而

$$(u(x), \overline{v(x)}) = \lim_{y\to 0} \int_{-\infty}^{\infty} \cdots \int u(x, y)\overline{v(x)}dx.$$

(2) 其他的情况可引进

$$u(x, r) = \int_0^\infty \frac{r^v dv}{\varphi(v)} \int_{\tau < v} a(t) e^{-itx'} dt \Big/ \int_0^\infty \frac{r^v dv}{\varphi(v)},$$

而

$$(u(x), \overline{v(x)}) = \lim_{r\to\infty} \int_{-\infty}^{\infty} \cdots \int u(x, r)\overline{v(x)}dx_1 \cdots dx_n.$$

这方法似乎比 Schwarz 的方法有显著的优点.

(3) 我们也可以用任一椭圆型偏微分方程来代替 Laplace 方程, 如此对应一个域内的解, 我们在边界上可以定义一广义函数理论. 当然还可以推得更广些, 如我们所研究过的典型域的调和函数与特征流形上的 Fourier 分析等等.

附录　求　和　法

定理 1　命 $q_v(r)$ 是一函数贯, 在一以 r_0 为右端的区间内定义, 并假定有以下的性质:

(i) $q_v(r) \geqslant 0$;

(ii) $\sum_{v=0}^{\infty} q_v(r) = 1,\ r < r_0$;

并对任一 v,

(iii) $\lim_{r \to r_0} q_v(r) = 0$,

则由 $s_n \to s$ 可推得

$$\lim_{r \to r_0} \sum_{v=0}^{\infty} = q_v(r) s_v = s.$$

证　并不失去普遍性可以假定 $s = 0$, 对任与 $\varepsilon > 0$, 我们有 M 使当 $v \geqslant M$ 时, $|s_v| < \varepsilon$, 并可假定对所有的 $v, |s_v| \leqslant B$, 则

$$\begin{aligned}\left|\sum_{v=0}^{\infty} q_v(r) s_v\right| &\leqslant B \sum_{v=0}^{M} q_v(r) + \varepsilon \sum_{v=M+1}^{\infty} q_v(r) \\ &\leqslant B \sum_{v=0}^{M} q_v(r) + \varepsilon.\end{aligned}$$

当 $r \to r_0$ 时, 右边趋于 ε. 故得所证.

现在取两个重要特例:

(1) 取 $r_0 = 1, q_v(r) = (1-r)r^v$, 则条件 (i), (ii), (iii) 都适合, 所以当 $r \to 1$ 时

$$\sum_{v=0}^{\infty} (1-r) r^v s_v = \sum_{v=0}^{\infty} (s_v - s_{v-i}) r^v \to \lim_{v \to \infty} s_v.$$

这就是

定理 2 (Abel)　若

$$\sum_{v=0}^{\infty} a_v$$

收敛于 s, 则当 $r \to 1$ 时

$$\sum_{v=0}^{\infty} a_v r^v$$

也趋于 s.

(2) 假定

$$\sum_{v=0}^{\infty} p_v r^v, \quad p_v \geqslant 0$$

是一处处收敛的幂级数. 命

$$q_v(r) = p_v r^v \Big/ \sum_{v=0}^{\infty} p_v r^v,$$

则 (i), (ii) 显然适合. 由于

$$\lim_{r\to\infty} q_v(r) = \lim_{r\to\infty} \frac{v p_v r^{v-1}}{\sum\limits_{v=1}^{\infty} v p_v r^{v-1}} = \cdots = 0,$$

由此得出

定理 3 (Borel) 若

$$\sum_{v=0}^{\infty} p_v r^v, \quad p_v \geqslant 0$$

是一处处收敛的幂级数. 如果 $s_v \to s$, 则

$$\lim_{r\to\infty} \frac{\sum\limits_{v=0}^{\infty} p_v r^v s_v}{\sum\limits_{v=0}^{\infty} p_v r^v} = s.$$

定理 1 还有其类似的积分定理.

定理 4 命 $q(r,\theta)$ 是一函数在 $0 \leqslant \theta \leqslant 2\pi$ 及 $0 \leqslant r < 1$ 中定义且有次之性质:

(i) $q(r,\theta) \geqslant 0$;

(ii) $\displaystyle\int_0^{2\pi} q(r,\theta)d\theta = 1$

及对任一 $\varepsilon > 0$,

$$\lim_{r\to 1} \int_{|\theta|\geqslant\varepsilon} q(r,\theta)d\theta = 0.$$

如此, 若当 $\theta \to \pm 0$ 时, $f(\theta) \to s$, 则

$$\lim_{r\to 1} \int_0^{2\pi} q(r,\theta) f(\theta)d\theta = s$$

(此处假定式 $f(\theta)$ 是囿变函数).

华罗庚文集·多复变函数论卷Ⅱ·下部

On the Theory of Automorphic Functions of a Matrix Variable I——Geometrical Basis*

(*Dedicated to Professor K. L. Hiong, Chancellor of National Yunnan University, on his fiftieth birthday*)

The present paper is a revised form of another manuscript which the author had previously submitted for publication. The revision was necessary because the original mannscript contained some results (found independently by the author in some research begun in 1941) that have been recently published in Prof. C. L. Siegel's paper on Symplectic Geometry.① It is the aim of this paper to give a brief account of those results which are interfluent with Siegel's contributions. The remaining part of the author's research will be given later separately.

The paper is divided into two parts: the frst part (1-7) is algebraic in nature and gives a very brief description of the main theory with which the author deals. In the second part (8-10) the author proves that the spaces which play the important roles in the theory of analytic mappings have nonpositive Riemannian curvature. Thus the geometries under consideration are sufficiently regular, and the development (in broad-line) of the theory of automorphic functions presents no serious difficulties.

The situation of the problem is well described by a statement due to Poincaré:La géomètrie non-euclidienne est la clef véritable du probleme qui nous occupe. *Acta.*

Note: Because of the poor mail service between the U. S. and China, a number of minor changes in this paper have been made here, with the consent of the editors, by Prof. Hua's friend Dr. Hsio-Fn Tuan and Prof. C. L. Siegel.

* Received September 10, 1943. Reprinted from the *American Journal of Mathemaics*, 1944, 66: 470–488.

① C. L. Siegel. Symplectic Geometry. *American Journal of Mathcmatics,* 1943, 65: 1–86. Auother important reference is:

C. L. Siegel. Einfürung in die Theorie der Modulfunktionen n-ten Grades. *Math Annalen,* 1939, 116: 617–657.

The author is greatly indebted to Prof H. Weyl for sending him a copy of Siegel's paper on Symplectic Geometry. The author would like also to express his thanks to Prof. P. C. Tang and Prof. S. S. Chern, for each sent to him one of the following two important references:

G. Giraud. *Lecons sur les fouclions automophes.* Paris: Gautlrier-Villars. 1920:

E. Cartan. Sur les domaines bornés homogènes de Fespace de n variables complexes. *Hamb Abh,* 1935, 11: 116–162.

Math, 1923, 39: 100.

1. Groups

Throughout the paper, capital Latin letters denote $n \times n$ matrices with complex elements unless the contrary is stated. A' denotes the transposed matrix of A and $\bar{A}$ denotes the conjugate complex matrix of A. I denotes the unit matrix and O denotes the zero matrix.

We use the notations

$$\mathfrak{F} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}, \quad \mathfrak{F}_1 = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$$

and

$$\mathfrak{I} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$$

Let

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We shall consider three types of matrices $\mathfrak{T}$:

(i) Those $\mathfrak{T}$ satisfying

$$\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$$

are called *symplectic*. The condition may be written as

$$AB' = BA', \quad CD' = DC', \quad AD' - BC' = I;$$

(ii) Those $\mathfrak{T}$ satisfying

$$\mathfrak{T}\mathfrak{F}_1\mathfrak{T}' = \mathfrak{F}_1$$

are called *orthogonal,* or more definitely, $\mathfrak{F}_1$-*orthogonal.* The condition may be written as

$$AB' = -BA', \quad CD' = -DC', \quad AD' + BC' = I;$$

(iii) Those $\mathfrak{T}$ satisfying

$$\bar{\mathfrak{T}}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$$

are called *conjunctive-symplectic*. The condition may be written as

$$\bar{A}B' = \bar{B}A', \quad \bar{C}D' = \bar{D}C', \quad \bar{A}D' - \bar{B}C' = I.$$

Remark Apparently we have a fourth type of matrices $\mathfrak{T}$ satisfying

$$\overline{\mathfrak{T}}\mathfrak{F}_1\mathfrak{T}' = \mathfrak{F}_1.$$

Since $\mathfrak{F}$ and $\mathfrak{F}_1$ are conjunctive the fourth type coincides with (iii).

Theorem 1 *Each type of matrices form a group with respect to multiplication.*

Definition The group formed by symplectic matrices is called the *symplectic group.* Similarly we define the $\mathfrak{F}_1$-orthogonal group and the *conjunctive-symplectic group.*

2. Spaces analogous to the projective space (a detailed study will be given elsewhere later)

Definition 1 A pair of matrices (Z_1, Z_2) is said to be *symmetric* or *skew-symmetric* if we have

$$(Z_1, Z_2)\mathfrak{F}(Z_1, Z_2)' = O$$

or

$$(Z_1, Z_2)\mathfrak{F}_1(Z_1, Z_2)' = O$$

respectively.

(Certainly we might define Hermitian pairs, but this would be of no interest in the study of automorphic functions).

Definition 2 A pair of matrices (Z_1, Z_2) is said to be *non-singular,* if the rank of the $n \times 2n$ matrix (Z_1, Z_2) is equal to n.

Definition 3 *A symplectic transformation* is defined by

$$(W_1, W_2) = Q(Z_1, Z_2)\mathfrak{T},$$

where Q is non-singular and $\mathfrak{T}$ is symplectic. Similarly, we define $\mathfrak{F}_1$*-orthognal and conjunctive-symplectic transformations.*

Theorem 2 *A symplectic transformation carries a non-singular symmetric pair of matrices into a non-singular symmetric pair. An* $\mathfrak{F}_1$*-orthogonal transformation carries a non-singular skew-symmetric pair of matrices into a non-singular skew-symmetric pair.*

Proof(for the symplectic case)

$$\begin{aligned}(W_1, W_2)\mathfrak{F}(W_1, W_2)' &= Q(Z_1, Z_2)\mathfrak{T}\mathfrak{F}\mathfrak{T}'(Z_1, Z_2)'Q' \\ &= Q(Z_1, Z_2)\mathfrak{F}(Z_1, Z_2)'Q' = O.\end{aligned}$$

Thus we may take a non-singular symmetric pair (Z_1, Z_2)② as a point of the space and the symplectic group as the group of motions of the space. Then we obtain a geometry analogous to the projective geometry. A similar consideration holds for skew-symmetric pairs. A detailed treatment of independent interest will be given elsewhere.

We can also take a non-singular pair (Z_1, Z_2) as a point of our space and the conjunctive-symplectic group as the group of motions of the space. Then we also obtain a type of geometry.

We now indicate a general treatment which will be described for the symplectic case only.

Let $\mathfrak{H}$ be a Hermitian matrix. We define the symmetric pairs making

$$(\overline{Z_1, Z_2})\mathfrak{H}(Z_1, Z_2)'$$

positive definite to be a space③. The group of motions of the space is the subgroup of the symplectic group leaving $\mathfrak{H}$ invariant. Thus we establish a geometry analogous to non-euclidean geometry.

Thus the symplectic classification of Hermitian matrices is of the first importance. After the classification and the study of the structure of the group of automorphisms we arrive at the conclusion that there are three types of geometries of fundamental importance, namely those with

$$\mathfrak{H} = \begin{pmatrix} H & O \\ O & H \end{pmatrix}, \tag{1}$$

where H is a diagonal matrix $[1,\cdots,1-1,\cdots,-1]$; those with

$$\mathfrak{H} = \begin{pmatrix} H & O \\ O & O \end{pmatrix}; \tag{2}$$

and those with

$$\mathfrak{H} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}. \tag{3}$$

Correspondingly, we define the related geometries to be elliptic with signature H, parabolic with signature H and hyperbolic, respectively.

② Identify (Z_1, Z_2) with (QZ_1, QZ_2) for any non-singular Q

③ It is defined to be a hypercircle in the "projective" space.

In this paper, we consider only the hyperbolic geometry. More definitely, all non-singular pairs (Z_1, Z_2) of matrices, making

$$(\overline{Z_1, Z_2}) \begin{pmatrix} I & O \\ O & -I \end{pmatrix} (Z_1, Z_2)'$$

positive definite, form a hyperbolic space. The symplectic transformations with matrix $\mathfrak{T}$ satisfying

$$\overline{\mathfrak{T}} \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \mathfrak{T}' = \rho \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \quad \rho = \pm 1$$

form the group of motions of the space.

Analogously, the non-singular skew-symmetric pairs (Z_1, Z_2) of matrices making

$$(\overline{Z}_1, \overline{Z}_2) \begin{pmatrix} I & O \\ O & -I \end{pmatrix} (Z_1, Z_2)'$$

positive definite form a space. The $\mathfrak{F}_1$-orthogonal transformations $\mathfrak{T}$ satisfying

$$\overline{\mathfrak{T}} \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \mathfrak{T}' = \rho \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$$

form the group of motions of the space.

There is a distinction between the symplectic and the conjunctive-symplectic cases, since the introduction of a "hypercircle" is not necessary in the latter case. The non-singular pairs of matrices (Z_1, Z_2) making

$$(\overline{Z_1, Z_2}) \begin{pmatrix} O & I \\ -I & O \end{pmatrix} (Z_1, Z_2)'$$

positive definite form the space, and the conjunctive-symplectic group is the group of motions of the space.

Notice that the transformation

$$(W_1, W_2) = Q(Z_1, Z_2) \begin{pmatrix} iI/\sqrt{2} & iI/\sqrt{2} \\ -iI/\sqrt{2} & iI/\sqrt{2} \end{pmatrix}$$

carries the space of points (Z_1, Z_2) such that

$$\text{“}(\overline{Z_1, Z_2}) \begin{pmatrix} I & O \\ O & -I \end{pmatrix} (Z_1, Z_2)' \text{ is positive definite ”}$$

into the space of points (W_1, W_2) such that

$$\text{“}(\overline{W_1, W_2}) \begin{pmatrix} O & I \\ I & O \end{pmatrix} (W_1, W_2)' \text{ is positive definite ”}$$

3. An extension of the conjunctive-symplectic group

We now consider the conjunctive-symplectic case with

$$\mathfrak{H} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

It is clear that there is no essential reason to restrict $\mathfrak{H}$ to be a Hermitian matrix with signature (n, n), except to have an analogy with the symplectic case. Thus we may extend the conjunctive-symplectic case much further. The procedure is as follows:

Let

$$\mathfrak{H} = \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix}$$

The points of the space are then given by matrices

$$(Z_1^{(n)}, \quad Z_2^{(n,m)})④$$

such that

$$\overline{(Z_1^{(n)}, Z_2^{(n,m)})}\mathfrak{H}(Z_1^{(n)}, Z_2^{(n,m)})'$$

is positive definite The group of motions consists of the transformations

$$(W_1^{(n)}, W_2^{(n,m)}) = Q(Z_1^{(n)}, Z_2^{(n,m)})\mathfrak{T}^{(n+m)},$$

where

$$\overline{\mathfrak{T}}^{(n+m)} \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix} \mathfrak{T}' = \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix}.$$

Writing

$$\mathfrak{T} = \begin{pmatrix} A^{(n)} & B^{(n,m)} \\ C^{(m,n)} & D^{(m)} \end{pmatrix}.$$

we have the conditions:

$$\bar{A}A' - \bar{B}B' = I, \quad \bar{A}C' = \bar{B}D', \quad \bar{C}C' - \bar{D}D' = I.$$

The group so obtained is called the conjunctive group of signature (n, m).

④ $Z^{(n)}$ denotes an $n \times n$ matrix: $Z^{(n,m)}$ denotes an $n \times m$ matrix.

4. Non-homogeneous coördinates

Let (W_1, W_2) and (Z_1, Z_2) be two symmetric (or skew) pairs connected by

$$(W_1, W_2) = Q(Z_1, Z_2)\mathfrak{T}.$$

If W_1 and Z_1 are both non-singular, let

$$W = -W_1^{-1}W_2, \quad Z = -Z_1^{-1}Z_2;$$

then W and Z are symmetric (or skew) matrices connected by

$$W = (-A + ZC)^{-1}(B - ZD),$$

i.e.,

$$Z = (AW + B)(CW + D)^{-1}.$$

Thus a non-singular symmetric pair of matrices (W_1, W_2) may be considered as the homogeneous coordinates of a symmetric (or skew) matrix W.

Now in non-homogeneous coördinates, the geometries take the following forms:

(i) The space is formed by the symmetric matrices Z satisfying[5]

$$I - Z\overline{Z} > 0.$$

The group of motions is given by

$$W = (AZ + B)(CZ + D)^{-1}$$

and

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic satisfying

$$\overline{\mathfrak{T}} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \mathfrak{T}' = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

(ii) The space is formed by the skew-symmetric matrices Z satisfying

$$I + Z\overline{Z} > 0.$$

⑤ "> 0" means "being positive definite."

The group of motions is given by

$$W = (AZ + B)(CZ + D)^{-1},$$

where $\mathfrak{T}$ is $\mathfrak{F}$-orthogonal leaving $I + Z\bar{Z} > 0$ invariant.

(iii) The space is formed by the $n \times m$ matrices Z satisfying

$$I - Z\bar{Z}' > 0.$$

The group of motions is the conjunctive group of signature (n,m).

As we transform $\begin{pmatrix} I & O \\ O & -I \end{pmatrix}$ into $\begin{pmatrix} O & I \\ I & O \end{pmatrix}$, we find that the symplectic and the conjunctive-symplectic cases have also the following equivalent expressions.

For the symplectic case, the space is formed by the symmetric matrices

$$Z = X + iY$$

with Y positive definite, and the group of motions can be easily verified to be the real symplectic group.

For the conjunctive-symplectic case, we define

$$\frac{Z' + \overline{Z}}{2}, \quad \frac{Z' - \overline{Z}}{2i}$$

to be the virtual real and imaginary parts of Z. The space is formed by the matrices Z with positive definite virtual imaginary parts. The group of motions is the conjunctive-symplectic group. Both correspond to the Poincaré half-plane.

Remark After laying the foundation of the present theory, the author found in Cartan's paper thar there are four general types (and two special types) of bounded symmetric spaces for analytic mappings. They are the previous types (i), (ii) and (iii), and a type studied by G. Giraud (in 1920) with the hyperabelian group. Thus the present treatment may be considered as complete in a certain sense.

5. Metrization of the space

Let $\mathfrak{H}$ be a Hermitian matrix

$$\mathfrak{H} = \begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}.$$

Let $\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a symplectic matrix satisfying

$$\overline{\mathfrak{T}}\mathfrak{H}\mathfrak{T}' = \mathfrak{H}.$$

and set

$$H(Z) = \overline{Z}H_2Z + \overline{Z}L + \bar{L}'Z + H_1;\ Z = (AW - B)(-CW + D)^{-1}.$$

Then

$$W = (D'Z + B')(C'Z + A')^{-1} = (ZC + A)^{-1}(ZD + B).$$

Therefore

$$\begin{aligned} H(W) =&(\overline{ZC + A})^{-1}\{(\bar{Z}\bar{D} + \bar{B})H_2(D'Z + B') \\ &+ (\bar{Z}\bar{D} + \bar{B})L(C'Z + A') + (\bar{Z}\bar{C} + \bar{A})\bar{L}'(D'Z + B') \\ &+ (\bar{Z}\bar{C} + \bar{A})H_1(C'Z + A')\}(C'Z + A')^{-1} \\ =&(\overline{ZC + A})^{-1}H(Z)(C'Z + A')^{-1}. \end{aligned}$$

Further

$$\begin{aligned} dW &= (ZC + A)^{-1}dZD - (ZC + A)^{-1}dZC(ZC + A)^{-1}(ZD + B) \\ &= (ZC + A)^{-1}dZ(ZC + A)'^{-1}. \end{aligned}$$

Therefore we have

Theorem 3 *The characteristic equation of the matrix*

$$dZ(H(Z))^{-1}d\overline{Z}(\overline{H(Z)})^{-1}$$

is invariant under the group of automorphisms of the hypercircle H(Z). In particular, we have

$$\sigma((H(Z))^{-1}d\overline{Z}(\overline{H(Z)})^{-1}dZ)$$

as an invariant quadratic differential form under the group, where $\sigma(X)$ denotes the trace of the matrix X.

For the case corresponding to the Poincaré half-plane, we have that the quadratic differential form

$$\sigma(Y^{-1}dZY^{-1}d\overline{Z})$$

is invariant under all real symplectic transformations.

A similar result holds for the $\mathfrak{F}$-orthogonal case.

Theorem 4 *For the conjunctive group with signature (n,m), we have the invariant quadratic differential form*

$$\sigma((I - Z\bar{Z}')^{-1}dZ(I - \bar{Z}'Z)^{-1}d\bar{Z}'),$$

where

$$Z = Z^{(n,m)}.$$

Proof Let

$$\mathfrak{T}^{(m+n)} = \begin{pmatrix} A^{(n,n)}, & B^{(n,m)} \\ C^{(m,n)}, & D^{(m,n)} \end{pmatrix}$$

be a matrix satisfying the condition

$$\bar{\mathfrak{T}}\begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix}\mathfrak{T}' = \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix}.$$

This condition may also be written as

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}\begin{pmatrix} A' & -C' \\ -B' & D' \end{pmatrix} = I^{(m+n)}.$$

Consequently

$$\begin{pmatrix} A' & -C' \\ -B' & D' \end{pmatrix}\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = I^{(m+n)};$$

i.e.,

$$A'\bar{A} - C'\bar{C} = I, \quad A'\bar{B} = C'\bar{D}, \quad -B'\bar{B} + D'\bar{D} = I.$$

On setting

$$W^{(n,m)} = (AZ^{(n,m)} + B)(CZ + D)^{-1},$$

we have

$$W = (Z\bar{B}' + \bar{A}')^{-1}(Z\bar{D}' + \bar{C}').$$

Then

$$\begin{aligned} I - W\bar{W}' =& I - (Z\bar{B}' + \bar{A}')^{-1}(Z\bar{D}' + \bar{C}')(D\bar{Z}' + C)(B\bar{Z}' + A)^{-1} \\ =&(Z\bar{B}' + \bar{A}')^{-1}((Z\bar{B}' + \bar{A}')(B\bar{Z}' + A) \\ &- (Z\bar{D}' + \bar{C}')(D\bar{Z}' + C))(B\bar{Z}' + A)^{-1} \\ =&(Z\bar{B}' + \bar{A}')^{-1}(I - Z\bar{Z}')(B\bar{Z}' + A)^{-1}. \end{aligned}$$

Furthermore

$$I - \bar{W}'W = I - (\overline{CZ+D})'^{-1}(\overline{AZ+B})'(AZ+B)(CZ+D)^{-1}$$
$$= (\overline{CZ+D})'^{-1}(I - \bar{Z}'Z)(CZ+D)^{-1}.$$

Finally, we have

$$dW = AdZ(CZ+D)^{-1} - (AZ+B)(CZ+D)^{-1}CdZ(CZ+D)^{-1}$$
$$= (A - (Z\bar{B}' + \bar{A}')^{-1}(Z\bar{D}' + \bar{C}')C)dZ(CZ+D)^{-1}$$
$$= (Z\bar{B}' + \bar{A}')^{-1}dZ(CZ+D)^{-1}.$$

Combining all these results, we have

$$(I - W\bar{W}')^{-1}dW(I - \bar{W}'W)^{-1}d\bar{W}'$$
$$=(B\bar{Z}' + A)(I - Z\bar{Z}')^{-1}dZ(I - \bar{Z}'Z)^{-1}d\bar{Z}'(B\bar{Z}' + A)^{-1}.$$

This completes the proof.

6. Unitary equivalence

Lemma *Given a unitary matrix U, there exists a matrix V such that*

(a) $V^2 = U, V$: *unitary*,

and

(b) *if* $U'A = AU$, *then* $V'A = AV$.

Proof There exists a unitary matrix Γ such that

$$\Gamma^{-1}U\Gamma = D,$$

where D is a diagonal matrix $[e^{i\theta_1}, \cdots, e^{i\theta_n}]$ and $0 \leqslant \theta_v < 2_\pi$. Let

$$V = \Gamma D_{\frac{1}{2}}\Gamma^{-1},$$

where $D_{\frac{1}{2}} = [e^{\frac{1}{2}i\theta_1}, \cdots, e^{\frac{1}{2}\theta_n}]$. Now we shall verify that V possesses the required properties. (a) is evident. For (b): If $U'A = AU$, then

$$\Gamma'^{-1}D\Gamma'A = A\Gamma D\Gamma^{-1}.$$

i.e.,

$$D\Gamma'A\Gamma = \Gamma'A\Gamma D.$$

Thus D is commutative with $\Gamma' A\Gamma$; then so is $D_{\frac{1}{2}}$, i.e.,

$$D_{\frac{1}{2}}\Gamma' A\Gamma = \Gamma' A\Gamma D_{\frac{1}{2}}.$$

Consequently

$$V'A = AV.$$

Theorem 5 *Let Z be a non-singular symmetric matrix with complex elements; then there exists a unitary matrix U such that*

$$UZU' = [\mu_1, \cdots, \mu_n],$$

where $\mu_1, \cdots, \mu_n$ are the positive square roots of the characteristic roots of $Z\overline{Z}$.

Proof Since $Z\overline{Z}$ is a positive definite Hermitian matrix, we have a unitary matrix V such that

$$VZ\bar{Z}\bar{V}' = [\mu_1^2, \cdots, \mu_n^2], \quad \mu_v > 0;$$

i.e.,

(1) $W\bar{W} = [\mu_1^2, \cdots, \mu_n^2]$, where $W = VZV'$. Evidently, $W_0 = [\mu_1, \cdots, \mu_n]$ is a solution of (1).

Now

$$W\bar{W} = W_0\bar{W}_0, \text{ i.e., } \quad (W_0^{-1}W)(\overline{W_0^{-1}W})' = I,$$

i.e., $W_0^{-1}W$ is unitary, say U_0; then

$$W = W_0U_0.$$

Since W and W_0 are both symmetric, we have

$$W_0U_0 = U_0'W_0.$$

By the lemma, we have a unitary matrix U such that $U^2 = U_0$ and $W_0U = U'W_0$. Then

$$W = W_0U_0 = W_0U^2 = U'W_0U.$$

Thus

$$VZV' = U'W_0U,$$

and we have proved the theorem.

Theorem 6 *Let Z be a non-singular skew-symmetric matrix, then $Z\bar{Z}$ is a negative definite Hermitian matrix and its characteristic polynomial is a perfect square.*

Proof Evidently, we have

$$Z\bar{Z} = -Z\bar{Z}',$$

hence $Z\bar{Z}$ is negative definite. Further

$$|Z\bar{Z} - \lambda I| = |\bar{Z}||Z - \lambda\bar{Z}^{-1}|.$$

Since $Z - \lambda\bar{Z}^{-1}$ is a skew-symmetric matrix, its determinant is a perfect square.

Theorem 7 *Let Z be a non-singular skew-symmetric matrix; then we have a unitary matrix U such that*

$$UZU' = \begin{pmatrix} O & d_1 \\ -d_1 & O \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} O & d_{n/2} \\ -d_{n/2} & O \end{pmatrix},$$

where $d_1^2, d_1^2, \cdots, d_{n/2}^2, d_{n/2}^2$ are the characteristic roots of $-Z\bar{Z}$.

Proof By Theorem 6, we have a unitary matrix V such that

$$VZ\bar{Z}'\bar{V}' = [d_1^2, d_1^2, \cdots, d_{n/2}^2, d_{n/2}^2].$$

Let $VZV' = W$. Clearly

$$W_0 = \begin{pmatrix} O & d_1 \\ -d_1 & O \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} O & d_{n/2} \\ -d_{n/2} & O \end{pmatrix}$$

satisfies

$$W\bar{W}' = W_0\bar{W}'_0.$$

The remaining part of the proof is the same as that of Theorem 5.

Remark Both Theorems 5 and 7 may be extended to the singular case without any essential difficulty.

7. Existence and uniqueness of the geodesic passing through two given points

Theorem 8 *Passing through any two points of the hyperbolic space with the symplectic group, there is one and only one geodesic.*

Before proceeding to prove this fundamental theorem, we shall require a theorem concerning the equivalence of point-pairs.

Theorem 9 *In the hyperbolic space with the symplectic group (in Poincaré's representation) any two points are equivalent to the two points iI and iD, where D is a diagonal matrix*

$$[d_1, \cdots, d_n], \quad d_v > 0.$$

Proof 1) The group is evidently transitive. Thus we can let one of the two points be iI.

2) The transformations leaving iI fixed are all of the form

$$Z_1 = (AZ + B)(-BZ + A)^{-1},$$

i.e.,

$$\frac{Z_1 - iI}{Z_1 + iI} = (A + Bi)\left(\frac{Z - iI}{Z + iI}\right)(A' + B'i).$$

Since

$$I = AA' + BB' = (A + Bi)(A - Bi)',$$

we have $A+Bi$ unitary, and then the theorem follows from Theorem 5.

Proof of Theorem 8 1) Without loss of generality, we may assume that the two points are

$$Ii \text{ and } Di. \quad D = [d_1, \cdots, d_n], \quad d_v > 0.$$

Both points belong to the subspace $X = O$. Since

$$\begin{aligned}\sigma(Y^{-1}d\bar{Z}Y^{-1}dZ) &= \sigma(Y^{-1}dXY^{-1}dX) + \sigma(Y^{-1}dYY^{-1}dY)\\ &\geqslant \sigma(Y^{-1}dYY^{-1}dY)\end{aligned}$$

and the equality holds only for $dX = O$, the geodesics connecting the two points all lie in the subspace $X = O$.

2)Further let

$$Y = M'\Lambda M$$

where

$$M = (\gamma_{ij}) \quad (\gamma_{ij} = 0 \quad \text{for} \quad i > j \quad \gamma_{ii} = 1).$$

and

$$\Lambda = [q_1, \cdots, q_n]$$

are obtained according to Jacobi's reduction of positive definite quadratic forms.

(Notice that the space with positive definite Y is mapped topologically into the space with $q_v > 0$ in the new variables (q, γ)).

Then we have

$$\begin{aligned}\dot{Y} &= \dot{M}'\Lambda M + M'\dot{\Lambda}M + M'\Lambda\dot{M},\\ Y^{-1}\dot{Y} &= M^{-1}\Lambda^{-1}\dot{\Lambda}M + M^{-1}\Lambda^{-1}M'^{-1}\dot{M}'\Lambda M + M^{-1}\dot{M},\end{aligned}$$

and

$$\sigma((Y^{-1}\dot{Y})^2) = \sigma((\Lambda^{-1}\dot{\Lambda})^2) + 2\sigma((M^{-1}\dot{M})^2) + 4\sigma(M^{-1}\Lambda^{-1}\dot{\Lambda}\dot{M}) + 2\sigma(M^{-1}\Lambda^{-1}M'^{-1}\dot{M}'\Lambda\dot{M}),$$

since

$$\sigma(AB) = \sigma(BA) \quad \text{and} \quad \sigma(A') = \sigma(A).$$

Further, let

$$M^{-1} = (n_{hk}), \quad n_{hk} = 0 \ \text{ for } \ h > k,$$

then

$$\sigma((M^{-1}\dot{M})^2) = \sum_{i,j,k,l} n_{ij}\dot{m}_{jk}n_{kl}\dot{m}_{li} = \sum_{i\leqslant j\leqslant k\leqslant l\leqslant i} n_{ij}\dot{m}_{jk}n_{kl}\dot{m}_{li} = \sum (n_{ii}\dot{m}_{ii})^2 = 0,$$

since $\dot{m}_{ii} = 0$; and

$$\sigma(M^{-1}\Lambda^{-1}\dot{\Lambda}\dot{M}) = \sum n_{ij}q_j^{-1}\dot{q}_j\dot{m}_{ji} = \sum_{i=j} n_{ij}q_j^{-1}\dot{q}_j\dot{m}_{ji} = 0.$$

Thus

$$\sigma(Y^{-1}\dot{Y}Y^{-1}\dot{Y}) = \sigma((\Lambda^{-1}\dot{\Lambda})(\Lambda^{-1}\dot{\Lambda})) + 2\sigma((\Lambda^{\frac{1}{2}}\dot{M}M^{-1}\Lambda^{-\frac{1}{2}})(\Lambda^{\frac{1}{2}}MM^{-1}\Lambda^{-\frac{1}{2}})' \geqslant \sigma((\Lambda^{-1}\dot{\Lambda})(\Lambda^{-1}\dot{\Lambda})'),$$

and the equality holds for $\dot{M} = O$. Therefore the geodesics connecting the two points lie in the subspace with $\dot{M} = O$, i.e., the subspace of real and diagonal Z.

The subspace of real and diagonal Z is Euclidean, since

$$\sigma((\Lambda^{-1}d\Lambda)(\Lambda^{-1}d\Lambda)' = \sum_{i=1}^{n}(d \log q_i)^2,$$

and we have proved the theorem.

Evidently, we may deduce

Theorem 10 *All geodesics are symplectic images of the curves*

$$Z = i[\lambda_1^s, \cdots, \lambda_n^s], \quad \lambda_v > 0$$

and $\sum_{v=1}^{n} \log^2 \lambda_v = 1.$

Theorem 11 *The equations of the geodesics of the space are given by*

$$d^2Z/ds^2 + i(dZ/ds)Y^{-1}dZ/ds = O.$$

The previous results give a sufficient indication of the algebraic treatment. We shall now give a general treatment which seems to be a "direct" attack.

8. A general type of Riemannian geometry

Let

$$S = S^{(n,m)} = (s_{ij}); \quad T = T^{(m,n)} = (t_{ij}).$$

We consider the geometry with the Riemannian metric

$$\sigma((I^{(n)} - ST)^{-1}dS(I^{(m)} - TS)^{-1}dT).$$

Let

$$\Lambda_{ij} = (\partial/\partial s_{ij})S.$$

Lemma *If*

$$\sigma(\Lambda_{ij}S') = 0$$

for all i and j, then $S = O^{(n,m)}$.

The lemma is evident.

Theorem 12 *The equations of the geodesics of the space are given by*

$$d^2T/ds^2 + 2(dT/ds)(I - ST)^{-1}(S - STS)(I - TS)^{-1}(dT/ds) = O,$$
$$d^2S/ds^2 + 2(dS/ds)(I - TS)^{-1}(T - TST)(I - ST)^{-1}(dS/ds) = O,$$

where I denotes, under evident circumstances, either $I^{(n)}$ or $I^{(m)}$.

Proof We have

$$\begin{aligned}\partial\sigma/\partial s_{ij} =&\sigma\{(I - ST)^{-1}\Lambda_{ij}T(I - ST)^{-1}(dS/ds)(I - TS)^{-1}(dT/ds)\\ &+ (I - ST)^{-1}(dS/ds)(I - TS)^{-1}T\Lambda_{ij}(I - TS)^{-1}(dT/ds)\}\\ =&\sigma(\Lambda_{ij}[T(I - ST)^{-1}(dS/ds)(I - TS)^{-1}(dT/ds)(I - ST)^{-1}\\ &+ (I - TS)^{-1}(dT/ds)(I - ST)^{-1}(dS/ds)(I - TS)^{-1}T]).\end{aligned}$$

Next

$$\begin{aligned}(d/ds)(\partial\sigma/\partial s_{ij}) =&(d/ds)(\sigma(\Lambda_{ij})(I - TS)^{-1}(dT/ds)(I - ST)^{-1})\\ =&\sigma(\Lambda_{ij}[(I - TS)^{-1}(TdS/ds + (dT/ds)S)\\ &\times (I - TS)^{-1}(dT/ds)(I - ST)^{-1}\\ &+ (I - TS)^{-1}(dT/ds)(I - ST)^{-1}(SdT/ds + (dS/ds)T)(I - ST)^{-1}\\ &+ (I - TS)^{-1}(d^2T/ds^2)(I - ST)^{-1}]).\end{aligned}$$

Thus

$$(d/ds)(\partial\sigma/\partial s_{ij}) - \partial\sigma/\partial s_{ij} = \sigma(\Lambda_{ij}(I - TS)^{-1}M(I - ST)^{-1}),$$

where

$$M = d^2T/ds^2 + (dT/ds)(I - ST)^{-1}SdT/ds + (dT/ds)S(I - TS)^{-1}(dT/ds),$$

since

$$T(I - ST)^{-1} = (I - TS)^{-1}T, \text{ etc.}$$

The equations of the geodesics are

$$(d/ds)(\partial\sigma/\partial s_{ij}) - \partial\sigma/\partial s_{ij} = 0$$

for all i and j. We have, then

$$d^2T/ds^2 + 2(dT/ds)(I - ST)^{-1}(S - STS)(I - TS)^{-1}(dT/ds) = O,$$

since

$$\begin{aligned}&(I - ST)^{-1}S + S(I - TS)^{-1}\\ =&(I - ST)^{-1}(S(I - TS) + (I - ST)S)(I - TS)^{-1}\\ =&2(I - ST)^{-1}(S - STS)(I - TS)^{-1}.\end{aligned}$$

Interchanging S and T, we have the other differential matrix-equation.

Theorem 13 *The Riemannian currature tensor of the space is given by*

$$\sigma(K(U,Y)K(U,Y) - K(U,Y)K(X,V) - K(X,Y)K(U,V) + K(X,V)K(X,V)),$$

where (U,V) and (X,Y) are two directions and

$$K(X,Y) = (I - ST)^{-1}X(I - TS)^{-1}Y.$$

Proof (The method is borrowed from Siegel's paper) We write

$$\begin{aligned}R(S,T) &= (I - TS)^{-1}T = T(I - ST)^{-1}\\ &= (I - TS)^{-1}(T - TST)(I - ST)^{-1}.\end{aligned}$$

Then

$$\begin{aligned}dR(S,T) &= (I - TS)^{-1}dT + (I - TS)^{-1}(TdS + dTS)(I - TS)^{-1}T\\ &= (I - TS)^{-1}TdS(I - TS)^{-1}T + (I - TS)^{-1}dT(I + S(I - TS)^{-1}T)\\ &= (I - TS)^{-1}(TdST + dT)(I - ST)^{-1}\end{aligned}$$

$$= R(S,T)dSR(S,T) + (I-TS)^{-1}dT(I-ST)^{-1}. \tag{1}$$

Now we define two covariant differentials $(\delta_1 U, \delta_1 V), (\delta_2 U, \delta_2 V)$ by

$$\delta_i U = -UR(S,T)\delta_i S - \delta_i SR(S,T)U, \quad i=1,2,$$

and $\delta_i V$ is defined similarly by interchanging S and T formally. Then

$$\begin{aligned}\delta_1\delta_2 U =& -\delta_1 UR(S,T)\delta_2 S - U\delta_1 R(S,T)\delta_2 S\\ &- \delta_2 S\delta_1 R(S,T)U - \delta_2 SR(S,T)\delta_1 U\\ =&(UR(S,T)\delta_1 S + \delta_1 SR(S,T)U)R(S,T)\delta_2 S\\ &- U(R(S,T)\delta_1 SR(S,T) + (I-TS)^{-1}\delta_1 T(I-ST)^{-1})\delta_2 S\\ &- \delta_2 S(R(S,T)\delta_1 SR(S,T) + (I-TS)^{-1}\delta_1 T(I-ST)^{-1})U\\ &+ \delta_2 SR(S,T)(UR(S,T)\delta_1 S + \delta_1 SR(S,T)U)\\ =&\delta_1 SR(S,T)UR(S,T)\delta_2 S + \delta_2 SR(S,T)UR(S,T)\delta_1 S\\ &- U(I-TS)^{-1}\delta_1 T(I-ST)^{-1}\delta_2 S\\ &- \delta_2 S(I-TS)^{-1}\delta_1 T(I-ST)^{-1}U.\end{aligned}$$

We have, then

$$\begin{aligned}U^* =&(\delta_1\delta_2 - \delta_2\delta_1)U\\ =&U(P(\delta_1 S, \delta_2 T) - P(\delta_2 S, \delta_1 T)) + (Q(\delta_1 S, \delta_2 T) - Q(\delta_2 S, \delta_1 T))U.\end{aligned}$$

where

$$\begin{aligned}P(A,B) &= (I-TS)^{-1}B(I-ST)^{-1}A,\\ Q(A,B) &= A(I-TS)^{-1}B(I-ST)^{-1}.\end{aligned}$$

Similarly, interchanging S and T, we have

$$\begin{aligned}V^* =&(\delta_1\delta_2 - \delta_2\delta_1)V\\ =&V(P^*(\delta_1 T, \delta_2 S) - P^*(\delta_2 T, \delta_1 S)) + (Q^*(\delta_1 T, \delta_2 S) - Q^*(\delta_2 T, \delta_1 S))V,\end{aligned}$$

where P^* and Q^* have obvious definitions.

We introduce a further covariant vector (X, Y). We have to evaluate

$$2R = \sigma(P(U^*, Y) + P(X, V^*)),$$

which is

$$\begin{aligned}
&\sigma\{(I-TS)^{-1}Y(I-ST)^{-1}U[P(\delta_1 S, \delta_2 T) - P(\delta_2 S, \delta_1 T)] \\
&+ U(I-TS)^{-1}Y(I-ST)^{-1}[Q(\delta_1 S. \delta_2 T) - Q(\delta_2 S, \delta_1 T)] \\
&+ (I-ST)^{-1}X(I-TS)^{-1}V[P^*(\delta_1 T, \delta_2 S) - P^*(\delta_2 T, \delta_1 S)] \\
&+ V(I-ST)^{-1}X(I-TS)^{-1}[Q^*(\delta_2 T, \delta_2 S) - Q^*(\delta_2 T, \delta_1 S)] \\
=&\sigma(P(U,Y)P(\delta_1 S, \delta_2 T) - P(U,Y)P(\delta_2 S, \delta_1 T) \\
&+ P(\delta_1 S, Y)P(U, \delta_2 T) - P(\delta_2 S, Y)P(U, \delta_1 T) \\
&+ P(\delta_2 S, V)P(X, \delta_1 T) - P(\delta_1 S, V)P(X, \delta_2 T) \\
&+ P(\delta_2 S, \delta_1 T)P(X, V) - P(\delta_1 S, \delta_2 T)P(X, V)).
\end{aligned}$$

Putting $\delta_1(S,T) = (U,V)$ and $\delta_2(S,T) = (X,Y)$,; we obtain the Riemannian curvature tensor

$$\begin{aligned}
R =&\sigma[P(U,Y)P(U,Y) - P(U,Y)P(X,V) \\
&- P(X,Y)P(U,Y) + P(X,V)P(X,V)].
\end{aligned}$$

Changing P into K, we obtain the result stated in the theorem

9. A specialization

In particular, we put

$$S = Z, \quad T = \bar{Z}' \quad V = \bar{U}', \quad X = \bar{Y}'$$

in the formula of Theorem 13. By the lemma below we then have

Theorem 14 *The Riemannian curvature tensor of the space with the metric*

$$\sigma(K(dZ, d\bar{Z}'))$$

is equal to

$$\sigma((K(U, \bar{X}') - K(X, \bar{U}'))^2),$$

where

$$K(A,B) = (I - Z\bar{Z}')^{-1}A(I - \bar{Z}'Z)^{-1}B.$$

Lemma *Under the hypothesis of the theorem*

$$\sigma(K(U,\bar{U}')K(X,\bar{X}')) = \sigma(K(X,\bar{U}')K(U,\bar{X}')).$$

Proof 1) The trace of the product of two Hermitian matrices A and B is real, since

$$\sigma(AB) = \sum_{r,s} a_{rs}b_{sr} = \sum \bar{a}_{sr}\bar{b}_{rs} = \sigma(\bar{A}\bar{B}).$$

2) We have

$$\begin{aligned}&\sigma(K(U,\bar{U}')K(X,\bar{X}'))\\=&\sigma((I-\bar{Z}'Z)^{-1}\bar{X}'(I-Z\bar{Z}')^{-1}U(I-\bar{Z}'Z)^{-1}\bar{U}'(I-Z\bar{Z}')^{-1}X),\end{aligned}$$

which is the trace of the product of two Hermitian matrices. Thus it is real.

3) Since

$$\sigma(\overline{(K(U,\bar{U}')K(X,\bar{X}'))'}) = \sigma(K(X,\bar{U}')K(U,\bar{X}')),$$

we have the theorem.

Theorem 15 *The Riemannian currratures of the three kinds of hyperbolic spaces are always non-positive.*

Proof We have

$$\begin{aligned}&K(U,\bar{X}') - K(X,\bar{U}')\\=&(I-Z\bar{Z}')^{-1}(U(I-\bar{Z}'Z)^{-1}\bar{X}' - X(I-\bar{Z}'Z)^{-1}\bar{U}')\\=&(I-Z\bar{Z}')^{-1}M,\end{aligned}$$

where M is skew-Hermitian. Then the Riemannian curvature tensor is

$$R = -\sigma((I-Z\bar{Z}')^{-1}M(I-Z\bar{Z}')^{-1}\bar{M}').$$

For those points making $I - Z\bar{Z}'$ positive definite, we have a matrix P such that

$$(I-Z\bar{Z}')^{-1} = P\bar{P}',$$

then

$$R = -\sigma(T\bar{T}'),$$

where $T = \bar{P}'MP$. Thus R is non-positive at all points for all directions of the space.

We conclude consequently that passing through two points there is one and only one geodesic and that from a point we can draw a geodesic perpendicular to a given geodesic, etc.⑥

It may also be proved that the spaces are Einstein spaces, i.e., their Ricci tensors are proportional to the fundamental tensors. According to a result due to Schouten and Struik, the spaces cannot be conformal to Euclidean spaces.

10. A concluding remark

Theorem 14 and its consequences and the properties of the groups given in Cartan's paper lead to a neat generalization of the theory of automorphic functions.

National Tsinghua Universiry, Institute of Mathematics, Acad Sinica

⑥ E. Cartan. *Lecons sur la géométrie des espaces de Riemann. Paris,* 1925, Note 3.

On the Theory of Automorphic Functions of a Matrix Variable II——the Classification of Hypercircles Under the Symplectic Group*

1. Introduction

The present paper is a continuation of the paper I with the same title①, which gives a brief account of the geometrical aspect of the theory.

Throughout the paper, capital Latin letters denote $n \times n$ matrices with complex elements unless the contrary is stated . A' denotes the transposed matrix of A and $\bar{A}$, the conjugate imaginary matrix of A. I denotes the unit matrix and O, the zero matrix.

We define a hypercircle to be the set of points (symmetric matrices) Z for which the Hermitian matrix

$$\bar{Z}H_1Z + LZ + \bar{Z}\bar{L}' + H_2$$

is positive definite, where H_1 and H_2 are Hermitian matrices.

The object of the present paper is to classify completely hypercircles under the (non-homogeneous) symplectic group $\mathfrak{G}$, which consists of all the symplectic transformations defined by:

$$Z_1 = (AZ + B)(CZ + D)^{-1}, \quad AB' = BA', \quad CD' = DC', \quad AD' - BC' = I.$$

The letter $\mathfrak{G}$ will be kept in this sense throughout the paper.

Our classification of hypercircles depends on the theory of pairs of Hermitian matrices. Because all the available treatments (or at least all the treatments available to the author in China, cf. 6) of the subject contain a mistake, we find it necessary to resume the theory.

Note: Because of the poor mail service between the U.S. and China, a number of minor changes in this paper have been made here, with the consent of the editors, by Prof. Hua's friend Dr. Hsio-Fu Tuan.

* Received April 21, 1943. Reprinted from the *American Journal of Mathematics*, 1944, 66: 531–563.

① This *Journal*, 1944, 66: 470–488.

2. Symmetric pairs of matrices

Let

$$\mathfrak{F} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

which is a $2n \times 2n$ skew symmetric matrix. This notation will be kept throughout.

Definition 1 A pair of matrices A and B is said to be *symmetric to each other*, or to form a *symmetric pair* (A, B), if $AB' = BA'$.

Clearly (A, B) *is a symmetric pair if and only if*

$$(A, B)\mathfrak{F}(A, B)' = O.$$

since the left hand side is equal to

$$(-B, A)(A, B)' = -BA' + AB'.$$

Definition 2 A pair of matrices (C, D) is said to be *conjugate to another pair of matrices* (A, B) if $AD' - BC' = I$.

According to this definition, the conjugate relation is skew in the two pairs: if (C, D) is conjugate to (A, B), then $-(A, B) = (-A, -B)$ is conjugate to (C, D). In the following we shall often speak of conjugate pairs, when the order of the pairs is immaterial.

Clearly (C, D) *is conjugate to* (A, B) *if and only if*

$$(A, B)\mathfrak{F}(C, D)' = I,$$

since the left hand side is exactly $AD' - BC'$.

Theorem 1 *The transformation*

$$Z_1 = (AZ + B)(CZ + D)^{-1}$$

with the matrix

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(in the following we often speak of the transformation $\mathfrak{T}$) *belongs to* $\mathfrak{G}$, *if and only if*

$$\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}.$$

Proof Since the left hand side of the equation to be proved is

$$\begin{pmatrix} AB' - BA', & AD' - BC' \\ CB' - DA' & CD' - DC' \end{pmatrix},$$

the result follows immediately.

Putting this result in another form, we have:

Theorem 2 *The transformation*

$$Z_1 = (AZ + B)(CZ + D)^{-1}$$

with the matrix

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $\mathfrak{G}$ *if and only if* (A, B) *and* (C, D) *are two symmetric pairs such that* (C, D) *is conjugate to* (A, B).

Theorem 3 *If* (A, B) *is a symmetric pair, then*

$$(A_1, B_1) = Q(A, B)\mathfrak{T}$$

is also a symmetric pair, where $\mathfrak{F}$ *is in* $\mathfrak{G}$.

Proof We have

$$\begin{aligned}(A_1, B_1)\mathfrak{F}(A_1, B_1)' &= Q(A, B)\mathfrak{T}\mathfrak{F}\mathfrak{T}'(A, B)'Q' \\ &= Q(A, B)\mathfrak{F}(A, B)'Q' = O.\end{aligned}$$

Theorem 4 *If* (A, B) *is conjugate to* (C, D), *and if*

$$(A_1, B_1) = Q(A, B)\mathfrak{T}, \quad (C_1, D_1) = Q'^{-1}(C, D)\mathfrak{T},$$

then (A_1, B_1) *is conjugate to* (C_1, D_1), *where* Q *is non-singular and* $\mathfrak{T}$ *is in* $\mathfrak{G}$.

Proof We have

$$(A_1, B_1)\mathfrak{F}(C_1, D_1)' = Q(A, B)\mathfrak{T}\mathfrak{F}\mathfrak{T}'(C, D)'Q^{-1} = I.$$

Definition 3 *Two symmetric pairs of matrices* (A_1, B_1) *and* (A, B) *are said to be equivalent if we have a non-singular matrix* Q *and a transformation* $\mathfrak{T}$ *of* $\mathfrak{G}$ *such that*

$$(A_1, B_1) = Q(A, B)\mathfrak{T}.$$

This relation will be denoted by

$$(A_1, B_1) \sim (A, B).$$

Theorem 5 *The relation "*$\sim$*" possesses the properlies: determinalion, reflexirity, symmetry and transitivity.*

Definition 4 A pair of matrices (A, B) is said to be *non-singular* if the matrix (A, B) is of rank n.

Theorem 6 *Any two non-singular symmetric pairs of matrices are equivalent.*

Proof It is sufficient to prove that

$$(A, B) \sim (I, O).$$

1) If A is non-singular, then $A^{-1}B = S$ is symmetric. Then

$$(A, B) = A(I, S) = A(I, O)\begin{pmatrix} I & S \\ O & I \end{pmatrix}.$$

The result follows, since

$$\begin{pmatrix} I & S \\ O & I \end{pmatrix}$$

belongs to $\mathfrak{G}$.

2) If A is singular, then we have two non-singular matrices P and Q such that

$$A_1 = PAQ = \begin{pmatrix} I^{(r)} & O \\ O & O^{(n-r)} \end{pmatrix}.$$

Let

$$(A_1, B_1) = P(A, B)\begin{pmatrix} Q & O \\ O & Q'^{-1} \end{pmatrix},$$

where

$$B_1 = PBQ'^{-1} = \begin{pmatrix} s^{(r)} & m \\ l & t^{(n-r)} \end{pmatrix}, \text{ say.}$$

Since

$$\begin{pmatrix} Q & O \\ O & Q'^{-1} \end{pmatrix}$$

belongs to $\mathfrak{G}$, (A_1, B_1) is a non-singular and symmetric pair. Consequently $s^{(r)}$ is symmetric and l is a null matrix.

Let

$$(A_2, B_2) = (A_1, B_1)\begin{pmatrix} I & -S \\ O & I \end{pmatrix},$$

where

$$S = \begin{pmatrix} s^{(r)} & O \\ O & I^{(n-r)} \end{pmatrix}.$$

Then

$$A_2 = A_1, \quad B_2 = -A_1 S + B_1 = \begin{pmatrix} O & m \\ O & t \end{pmatrix}.$$

Since (A_2, B_2) is non-singular, so also is t. Let

$$(A_3, B_3) = (A_2, B_2)\begin{pmatrix} I & O \\ I & I \end{pmatrix};$$

then

$$A_3 = A_2 + B_2 = \begin{pmatrix} I^{(r)} & m \\ O & t \end{pmatrix},$$

which is non-singular. By 1), we have

$$(A_3, B_3) \sim (I, O).$$

The result follows.

Theorem 7 *The subgroup which leaves a non-singular symmetric pair of matrices invariant is simply isomorphic to the group which consists of all transformations of the form*

$$Z_1 = Q'ZQ + S,$$

where Q is non-singular and S is symmetric.

Proof It is sufficient to consider the group which leaves (O, I) invariant. In fact, we have Q and $\mathfrak{T}$ such that

$$Q(A, B)\mathfrak{T} = (O, I).$$

Let Q_0 and $\mathfrak{T}_0$ be such that

$$Q_0(O, I)\mathfrak{T}_0 = (O, I).$$

Then

$$Q^{-1}Q_0Q(A, B)\mathfrak{T}\mathfrak{T}_0\mathfrak{T}^{-1} = (A, B).$$

The isomorphism of the group whose elements leave (O, I) invariant and the group whose elements leave (A, B) invariant is evident.

Let

$$Q(O, I)\mathfrak{T} = (O, I), \quad \mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then, we have

$$(QC, QD) = (O, I),$$

i.e., $C = O,\ D = Q^{-1}$. Then $A = Q'$ and $B = SQ^{-1}$.

The group is isomorphic to the group formed by the matrices

$$\begin{pmatrix} Q' & SQ^{-1} \\ O & Q^{-1} \end{pmatrix}.$$

The result is now evident.

Corollary *The transformations leaving (O, I) invariant are of the form*

$$\begin{pmatrix} Q' & SQ^{-1} \\ O & Q^{-1} \end{pmatrix},$$

where Q is non-singular and S is symmetric.

Theorem 8 *Given a non-singular symmetric pair of matrices (A, B), we have a non-singular symmetric pair of matrices (C, D) as its conjugate. The totality of all possible pairs (C, D) depends on $n(n+1)$ parameters.*

Proof 1) First we consider the case $(A, B) = (O, I)$. Let (C, D) be a pair satisfying our requirement, then

$$I = AD' - BC' = -C'.$$

Thus the conjugate pairs of (A, B) are

$$(-I, S),$$

where the S are symmetric. The theorem is true for $(A, B) = (O, I)$.

2) By Theorem 6, we have Q and $\mathfrak{T}$ such that

$$Q(A, B)\mathfrak{T} = (O, I).$$

We define (C, D) by $Q'^{-1}(C, D)\mathfrak{T} = (-I, S)$. Then (C, D) satisfies our requirement.

Further let Q_1 and $\mathfrak{T}_1$ be matrices satisfying also

$$Q_1(A, B)\mathfrak{T}_1 = (O, I).$$

Then, we have

$$(C, D) = Q_1'(-I, S_1)\mathfrak{T}_1^{-1}.$$

We shall now prove that this is equal to $Q'(-I,S)\mathfrak{T}^{-1}$. Since

$$Q_1Q^{-1}(O,I)\mathfrak{T}^{-1}\mathfrak{T}_1=(O,I)$$

by the corollary of Theorem 7, we have

$$\mathfrak{T}^{-1}\mathfrak{T}_1=\begin{pmatrix} Q'^{-1}Q_1' & S_2QQ_1^{-1} \\ O & QQ_1^{-1}\end{pmatrix}.$$

Then

$$\begin{aligned}Q_1'(-I,S_1)\mathfrak{T}_1^{-1}&=Q_1'(-I,S_1)\begin{pmatrix} Q'^{-1}Q_1' & S_2QQ_1^{-1} \\ O & QQ_1^{-1}\end{pmatrix}^{-1}\mathfrak{T}^{-1}\\ &=Q_1'(-I,S_1)\begin{pmatrix} Q_1'^{-1}Q' & -Q_1^{-1}Q'S_2 \\ O & Q_1Q^{-1}\end{pmatrix}\mathfrak{T}^{-1}\\ &=Q'(-I,Q'^{-1}Q_1'S_1Q_1Q^{-1}+S_2)\mathfrak{T}^{-1}.\end{aligned}$$

Hence we have always the same collection of pairs of matrices conjugate to (A,B).

3. Hypercircles

Definition 1　The transformation of symmetric pairs

$$(W_1,W_2)=Q(Z_1,Z_2)\mathfrak{T}$$

for a non-singular matrix Q and a transformation $\mathfrak{T}$ belonging to $\mathfrak{G}$ is called *a homogeneous representation of*

$$W=(AZ+B)(CZ+D)^{-1}.$$

The group so obtained is called *the group* $\mathfrak{G}_H$.

Definition 2　*A hypercircle* is defined by the set of points corresponding to symmetric matrices Z such that the Hermitian matrix

$$\bar{Z}H_1Z+LZ+\bar{Z}\bar{L}'+H_2$$

is positive definite, where H_1 and H_2 are Hermitian matrices. Or, in "homogeneous" coordinates, *a hypercircle* is defined by the set of points corresponding to symmetric pairs (W_1,W_2) such that the Hermitian matrix

$$\bar{W}_1H_1W_1'+\bar{W}_2LW_1'+\bar{W}_1\bar{L}'W_2'+\bar{W}_2H_2W_2'=(\overline{W_1},\bar{W}_2)\mathfrak{H}(W_1,W_2)'$$

is positive definite, where

$$\mathfrak{H}=\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2\end{pmatrix}.$$

$\mathfrak{H}$ is called *the matrix of the hypercircle.*

Remark $\mathfrak{H}$ is a general $2n \times 2n$ Hermitian matrix. Thus the following results may be interpreted purely algebraically without reference to hypercircles.

Theorem 9 *The transformation $(W_1, W_2) = Q(Z_1, Z_2)\mathfrak{T}$ carries a hypercircle with the matrix $\mathfrak{H}$ to a hypercircle with the matrix $\mathfrak{H}_1 = \bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'$.*

Proof Since

$$(\bar{W}_1, \bar{W}_2)\mathfrak{H}(W_1, W_2)' = \bar{Q}(\bar{Z}_1, \bar{Z}_2)\overline{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'(Z_1, Z_2)'Q',$$

the theorem follows.

Definition 3 If we have $\mathfrak{T}$ belonging to $\mathfrak{G}$ such that $\mathfrak{H}_1 = \overline{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'$, we say that $\mathfrak{H}_1$ and $\mathfrak{H}$ are *conjunctive under* $\mathfrak{G}$.

Evidently, "conjunctivity under $\mathfrak{G}$" possesses the properties: symmetry, reflexivity and transitivity. Naturally, this suggests the classification of hypercircles under $\mathfrak{G}$. This problem is by no means easy but it is solved completely. First of all, we introduce the following notion:

Definition 4 For a hypercircle with the matrix $\mathfrak{H}$, we define

$$\mathfrak{H}'\mathfrak{F}\mathfrak{H} = \begin{pmatrix} H_1'L - L'H_1, & H_1'H_2 - L'\bar{L}' \\ -H_2'H_1 + \bar{L}L, & \bar{L}H_2 - H_2'\bar{L}' \end{pmatrix}$$

to be *the discriminantal matrix of the hypercircle.* It will be denoted by $\mathfrak{D}(\mathfrak{H})$. Evidently $\mathfrak{D}(\mathfrak{H})$ is skew-symmetric.

Theorem 10 *If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjunctive under $\mathfrak{G}$, then $\mathfrak{D}(\mathfrak{H})_1$ and $\mathfrak{D}(\mathfrak{H})_2$ are congruent under $\mathfrak{G}$. More precisely, if $\overline{\mathfrak{T}}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{H}_2$, then*

$$\mathfrak{T}\mathfrak{D}(\mathfrak{H}_1)\mathfrak{T}' = \mathfrak{D}(\mathfrak{H}_2).$$

Proof Since

$$\mathfrak{D}(\mathfrak{H}_2) = \mathfrak{H}_2'\mathfrak{F}\mathfrak{H}_2 = \mathfrak{T}\mathfrak{H}_1'\overline{\mathfrak{T}}'\mathfrak{F}\overline{\mathfrak{T}}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{T}\mathfrak{H}_1'\mathfrak{F}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{T}\mathfrak{D}(\mathfrak{H}_1)\mathfrak{T}',$$

we have the result.

4. The canonical form of the discriminantal matrix

The problem of congruence of $\mathfrak{D}(\mathfrak{H}_1)$ and $\mathfrak{D}(\mathfrak{H}_2)$ under $\mathfrak{G}$ is equivalent to the problem of congruence of the pairs of skew symmetric matrices $(\mathfrak{D}(\mathfrak{H}_1), \mathfrak{F})$ and $(\mathfrak{D}(\mathfrak{H}_2), \mathfrak{F})$. The latter problem is solved in most treatises on elementray divisors. For the sake of completeness, the author quotes the following results:

Theorem 11 *Let $\mathfrak{B}$ and $\mathfrak{B}_1$ be two non-singular matrices. The pairs of skew symmetric matrices $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{A}_1, \mathfrak{B}_1)$ are congruent if and only if $\mathfrak{A} + \lambda\mathfrak{B}$ and $\mathfrak{A}_1 + \lambda\mathfrak{B}_1$ have the same invariant factors (or the same elementary divisors).*

(For the proof see, e. g., MacDuffee, *Theory of Matrices,* Theorems 35.4 and 30.1).

Theorem 12 *There exist pairs of skew symmetric matrices of degree $2n$, one of which is non-singular, having any given admissible invariant factors. More precisely, let*

$$h_{2i} = h_{2i-1} = g_i = (\lambda - \lambda_1)^{l_{i1}} \cdots (\lambda - \lambda_k)^{l_{ik}},$$

$$1 \leqslant i \leqslant n,\ l_{ij} \geqslant 0,\ 1 \leqslant j \leqslant k$$

be the given $2i$-th invariant factors (since in a skew symmetric matrix, the $2i$-th invariant factor is equal to the $(2i\text{-}1)$-th invariant factor), let g_i divide g_{i+1} and let $\sum l_{ij} = n$. We define τ_i to be the direct sum of matrices

$$\tau_i = \tau_{i1} \dot{+} \tau_{i2} \dot{+} \cdots \dot{+} \tau_{ik},$$

where

$$\tau_{ij} = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ 0 & 0 & \lambda_j & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j \end{pmatrix}$$

is of degree l_{ij} and $1 \leqslant j \leqslant h$. Further we define T by the direct sum

$$T = \tau_n \dot{+} \tau_{n-1} \dot{+} \cdots \dot{+} \tau_{n-t}.$$

Then the pair of skew symmetric matrices $(\mathfrak{E}, \mathfrak{F})$ with

$$\mathfrak{E} = \begin{pmatrix} O & T \\ -T' & O \end{pmatrix}, \quad \mathfrak{F} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$$

possesses the preassigned invariant factors.

Proof Let δ_i be the greatest common divisor of the i-rowed minors of $T - \lambda I$. Then, evidently,

$$\delta_i = g_1 \cdots g_i.$$

Further let d_i be the greates common divisor of the i-rowed minors of $\mathfrak{E} - \lambda\mathfrak{F}$. We need only fine the d_i for any even i. It is evident that d_{2i} is the g.c.d.of $\delta_i^2,\ \delta_{i-1}\delta_{i+1},\ \delta_{i-2}\delta_{i+2},\cdots$. Since

$$\delta_i^2 = g_1^2 \cdots g_i^2, \qquad \delta_{i-t}\delta_{i+t} = g_1^2 \cdots g_{i-t}^2 g_{i-t+1} \cdots g_{i+i},$$

and g_i divides g_{i+1}, we have δ_i^2 dividing $\delta_{i-t}\delta_{i+t}$. Thus $d_{2i} = \delta_i^2$. Then

$$h_{2i}h_{2i-1} = \frac{d_{2i}}{d_{2i-1}}\frac{d_{2i-1}}{d_{2i-2}} = \frac{\delta_{i-1}^2}{\delta_i^2}.$$

Since $h_{2i} = h_{2i-1}$, we have

$$h_{2i} = h_{2i-1} = g_i.$$

Consequently we have

Theorem 13 *Every discriminantal matrix is congruent under* $\mathfrak{G}$ *to a matrix of the form*

$$\begin{pmatrix} O & T \\ -T' & O \end{pmatrix},$$

where T *has the same meaning as given in Theorem* 12. *Consequently, every hypercircle is conjunctive under* $\mathfrak{G}$ *to a hypercircle with its discriminantal matrix of the prescribed form.*

Proof By Theorems 11 and 12, we have $\mathfrak{T}$ such that

$$\mathfrak{T}\mathfrak{D}(\mathfrak{H})\mathfrak{T}' = \begin{pmatrix} O & T \\ -T' & O \end{pmatrix}$$

and

$$\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}.$$

Let $\overline{\mathfrak{T}}^{-1}\mathfrak{H}\mathfrak{T}'^{-1} = \mathfrak{H}_1$; then $\mathfrak{H}_1$ has its discriminantal matrix in the described form.

5. Proof of the theorem that every hypercircle is conjunctive under $\mathfrak{G}$ to a "binomial" hypercircle

Theorem 14 *Every hypercircle is conjunctive under* $\mathfrak{G}$ *to a "binomial" hypercircle, or more precisely a hypercircle with the matrix*

$$\begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} h_1^{(r)} & O \\ O & O \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_2^{(r)} & O \\ O & O \end{pmatrix}, \quad \det(h_1^{(r)}) \neq 0.$$

Proof 1) the theorem is well-known for $n = 1$. By Theorem 13, it is sufficient to consider a hypercircle with the matrix

$$\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}$$

satisfying the condition

$$\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}' \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix} = \begin{pmatrix} O & * \\ * & O \end{pmatrix},$$

i.e., $H_1'L = L'H_1$ and $\bar{L}H_2 = H_2'\bar{L}'$.

If H_1 is non-singular, then

$$S = LH_1^{-1} = H_1'^{-1}L'$$

is symmetric. We have evidently that

$$\bar{Z}H_1Z + LZ + \bar{Z}\bar{L}' + H_2 = (\bar{Z} + S)H_1(Z + \bar{S}) + H_2 - SH_1\bar{S},$$

which is "binomial" in $Z + \bar{S}$. A similar result holds when H_2 is non-singular. The theorem is thus true for these cases.

2) Before going further, we require two lemmas.

Lemma 1 *Any symmetric matrix S may be expressed as $S = TT'$ where T is a matrix with zeros above the main diagonal* (well-known).

Lemma 2 *For any given matrix Q, we have a non-singular symmetric matrix S such that QS is symmetric.*

In fact, it is sufficient to find a non-singular solution of the matrix equation

$$QS = SQ',$$

where the symmetric matrix S is considered as an unknown. We have a non-singular matrix Γ such that

$$Q_1 = \Gamma^{-1}Q\Gamma$$

is of the Jordan's normal form, i.e. a direct sum of matrices of the form

$$J_i^{(j)} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

Then

$$Q_1 S_1 = S_1 Q_1',$$

where $\Gamma^{-1} S \Gamma'^{-1} = S_1$. Therefore, it is sufficient to find a solution of the equation with $Q = J_i^{(j)}$. Evidently

$$S^{(j)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a solution, since $S = S'$ and

$$J_i S = \begin{pmatrix} 0 & 0 & \cdots & 1 & \lambda_i \\ 0 & 0 & \cdots & \lambda_i & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_i & \cdots & 0 & 0 \\ \lambda_i & 0 & \cdots & 0 & 0 \end{pmatrix} = S J_i'.$$

3) We now consider all conjunctive hypercircles of $\mathfrak{H}$ under $\mathfrak{G}$ with "binomial" discriminantal matrices. Let $\mathfrak{H}$ be one of them with H_1 of the highest rank r. If $r = n$, this problem was solved in 1).

We have a non-singular matrix Q such that

$$\bar{Q}' H_1 Q = \begin{pmatrix} h^{(r)} & O \\ O & O \end{pmatrix}, \quad \det(h) \neq 0.$$

Since

$$\mathfrak{T} = \begin{pmatrix} Q' & O \\ O & Q^{-1} \end{pmatrix}$$

carries a hypercircle with "binomial" discriminantal matrix into one of the same natrue, we may assume, without loss of generality, that

$$H_1 = \begin{pmatrix} h & O \\ O & O \end{pmatrix}, \quad \det(h) \neq 0, \quad H_2 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

We shall now establish that $r \neq 0$ and that we may assume det $(g_{11}) \neq 0$.

Let

$$\mathfrak{D}(\mathfrak{H}) = \begin{pmatrix} O & K \\ -K' & O \end{pmatrix}$$

be its discriminantal matrix. We may transform $\mathfrak{H}$ in such a way that K is symmetric. In fact, by Lemmas 1 and 2, we have a symmetric matrix S such that (i) SK is symmetric and (ii) $S = TT'$ where T is a matrix with zeros above the main diagonal. Let

$$\mathfrak{T} = \begin{pmatrix} T' & O \\ O & T^{-1} \end{pmatrix},$$

which belongs to $\mathfrak{G}$. Then

$$\mathfrak{T}\mathfrak{D}(\mathfrak{H})\mathfrak{T}' = \begin{pmatrix} O & T'KT'^{-1} \\ -T^{-1}K'T & O \end{pmatrix},$$

where $T^{-1}K'T$ is symmetric, since

$$TT'K = K'TT' \quad \text{implies} \quad T^{-1}K'T = T'KT'^{-1}.$$

Further the first element in $\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'$ is equal to

$$\bar{T}'H_1T = \begin{pmatrix} * & * \\ O & * \end{pmatrix}\begin{pmatrix} h & O \\ O & O \end{pmatrix}\begin{pmatrix} * & O \\ * & * \end{pmatrix} = \begin{pmatrix} * & O \\ O & O \end{pmatrix}$$

of which the rank is still r. We may assume that

$$H_1 = \begin{pmatrix} h & O \\ O & O \end{pmatrix}, \quad \mathfrak{D}(\mathfrak{H}) = \begin{pmatrix} O & S \\ -S & O \end{pmatrix},$$

where S is symmetric.

Let ρ be any number. We have

$$\begin{pmatrix} I & O \\ \rho I & I \end{pmatrix}\begin{pmatrix} O & S \\ -S & O \end{pmatrix}\begin{pmatrix} I & \rho I \\ O & I \end{pmatrix} = \begin{pmatrix} O & S \\ -S & O \end{pmatrix}$$

and

$$\overline{\begin{pmatrix} I & O \\ \rho I & I \end{pmatrix}}\begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}\begin{pmatrix} I & \rho I \\ O & I \end{pmatrix} = \begin{pmatrix} H_1 & * \\ * & H_0 \end{pmatrix},$$

where

$$H_0 = |\bar{\rho}|^2 H_1 + \bar{\rho}\bar{L}' + \rho L + H_2.$$

It is evident that for ρ large. $r = 0$ if and only if $H_1 = L = H_2 = O$.

Let

$$H_0 = \begin{pmatrix} k_{11}^{(r)} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

then

$$k_{11} = |\rho|^2 h + \rho^* + \bar{\rho}^* + *.$$

For ρ large, k_{11} is non-singular.

4) Now we may assume that

$$H_1 = \begin{pmatrix} h^{(r)} & O \\ O & O \end{pmatrix}, \quad H_2 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix},$$

where det $(h) \neq 0$, det $(g_{11}) \neq 0$ and $r \neq 0$. Let

$$R = \begin{pmatrix} I & O \\ -g_{12}'\bar{g}_{11}^{-1} & I \end{pmatrix}, \quad \mathfrak{T} = \begin{pmatrix} R'^{-1} & O \\ O & R \end{pmatrix},$$

which belongs to $\mathfrak{G}$, such that

$$\overline{\mathfrak{T}}\mathfrak{H}\mathfrak{T}' = \begin{pmatrix} \bar{R}'^{-1} & O \\ O & \bar{R} \end{pmatrix} \begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix} \begin{pmatrix} R^{-1} & O \\ O & R' \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} h & O \\ O & O \end{pmatrix} & * \\ * & \begin{pmatrix} g & O \\ O & g_0 \end{pmatrix} \end{pmatrix},$$

where $g = g_{11}$. in fact

$$\bar{R}'H_1R^{-1} = \begin{pmatrix} I & * \\ O & I \end{pmatrix} \begin{pmatrix} h & O \\ O & O \end{pmatrix} \begin{pmatrix} I & O \\ * & I \end{pmatrix} = \begin{pmatrix} h & O \\ O & O \end{pmatrix},$$

$$\bar{R}H_2R' = \begin{pmatrix} I & O \\ -\bar{g}_{12}'g_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ \bar{g}_{21}' & g_{22} \end{pmatrix} \begin{pmatrix} I & -g_{11}^{-1}g_{12} \\ O & I \end{pmatrix} = \begin{pmatrix} g_{11} & O \\ O & g_0 \end{pmatrix}.$$

Since the rank of H_2 cannot be higher than r, $g_0 = O$.

Now we may write

$$H_1 = \begin{pmatrix} h & O \\ O & O \end{pmatrix}, \quad L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \quad H_2 = \begin{pmatrix} g & O \\ O & O \end{pmatrix},$$

where both h and g are non-singular. Since $H_1'L = L'H_1$ and $\bar{L}H_2 = H_2'\bar{L}'$, we have

$$h'l_{11} = l_{11}'h, \quad \bar{l}_{11}g = g'\bar{l}_{11}', \quad l_{12} = l_{21} = O.$$

As in 1), we may then assume that $l_{11} = O$ and det $(h) \neq 0$, but now g may be singular. By induction, we have $a_1^{(n-r)}$, $b_1^{(n-r)}$, $c_1^{(n-r)}$ and $d_1^{(n-r)}$ such that

$$\overline{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}} \begin{pmatrix} O & \bar{l}'_{22} \\ l_{22} & O \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}' = \begin{pmatrix} h_2^{(n-r)} & O \\ O & g_2^{(n-r)} \end{pmatrix},$$

$$\overline{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}} \begin{pmatrix} O & I^{(n-r)} \\ -I^{(n-r)} & O \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}' = \begin{pmatrix} O & I^{(n-r)} \\ -I^{(n-r)} & O \end{pmatrix},$$

and we may assume that the rank of h_2 is higher than that of g_2, for otherwise

$$\begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} h_2 & O \\ O & g_2 \end{pmatrix} \begin{pmatrix} O & I \\ I & O \end{pmatrix} = \begin{pmatrix} g_2 & O \\ O & h_2 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} I & O \\ O & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} O & O \\ O & b_1 \end{pmatrix}, \quad C = \begin{pmatrix} O & O \\ O & c_1 \end{pmatrix}, \quad D = \begin{pmatrix} I & O \\ O & d_1 \end{pmatrix},$$

then

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $\mathfrak{G}$ and H_1 of $\overline{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'$ is equal to $\begin{pmatrix} h & O \\ O & h_2 \end{pmatrix}$. Since its rank cannot be higher than r, we have $h_2 = O$. Consequently, $g_2 = O$. Then $l_{22} = O$. The result is now proved.

6. A lemma

For reasons explained in the Introduction. we fine it necessary first to discuss the theory of pairs of Hermitian matrices (6-9)② as a basis for the classification of hypercircles (10-16).

Lemma *If $q(x)$ is a polynomial, with real coefficients, which has no negative or zero root, then we have a real polynomial $\chi(x)$ such that $\chi^2(x) - x$ is divisible by $q(x)$.*

② Cf. Diekson. *Modern algebraic theories.* p.123, Theorem 10; MacDuffee. *Theory of matrices.* p. 63, Theorem 36.5; Turnbull and Aitken, *Theory of canonical matrices,* p.131, Lemma III; and Logsdon, *American Journal of Mathematrics*, 1922, 44: 247–260. An earlier paper of Muth, *Journ für Math.*, 1905, 128: 302-321 should be mentioned as one of importance in this connection.

Proof Let

$$q(x) = \lambda \prod_{i=1}^{R} (x - a_i)^{l_i} \prod_{j=1}^{t} ((x - \alpha_i)(x - \bar{\alpha}_i)),$$

where $a_i > 0$ and α_i is complex.

1) The theorem is true for

$$q(x) = (x - a)^l.$$

In fact the theorem is true for $l = 1$, for then $\chi(x) = \sqrt{a}$ is a solution. Suppose that we have a real polynomial $\chi_{l-1}(x)$ such that

$$\chi_{l-1}^2(x) - x = (x - a)^{l-1}\lambda(x), \quad l > 1,$$

where $\lambda(x)$ is a polynomial with real coefficients. Evidently $\chi_{l-1}(a) \neq 0$. Then

$$\chi_l(x) = \chi_{l-1}(x) - \frac{1}{2}\frac{\lambda(a)}{\chi_{l-1}(a)}(x - a)^{l-1}$$

satisfies our requirement, since

$$\begin{aligned} x_l^2(x) - x &\equiv \chi_{l-1}^2(x) - x - \frac{\lambda(a)}{\chi_{l-1}(a)}\chi_{l-1}(x)(x - a)^{l-1} \\ &\equiv \left(\lambda(x) - \frac{\lambda(a)}{\chi_{l-1}(a)}\chi_{l-1}(x)\right)(x - a^{l-1}) \\ &\equiv 0(\text{mod } (x - a)^l). \end{aligned}$$

2) The theorem is true for

$$q(x) = ((x - \alpha)(x - \bar{\alpha}))^l.$$

In fact

$$\chi(x) = \frac{1}{\sqrt{2|\alpha| + \alpha + \bar{\alpha}}}(x + |\alpha|)$$

satisfies our requirement for $l = 1$, since

$$\begin{aligned} \chi^2(x) - x &= \frac{1}{2|\alpha| + \alpha + \bar{\alpha}}(x^2 + 2|\alpha|x + |\alpha|^2) - x \\ &= \frac{1}{2|\alpha| + \alpha + \bar{\alpha}}(x - \alpha)(x - \bar{\alpha}) \\ &\equiv 0(\text{mod } (x - \alpha)(x - \bar{\alpha})) \end{aligned}$$

and $2|\alpha| + \alpha + \bar{\alpha} > 0$.

Let $\chi_{l-1}(x)$ be a real polynomial satisfying

$$\chi_{l-1}^2(x) - x = ((x-\alpha)(x-\bar{\alpha}))^{l-1}\lambda(x), \quad l > 1.$$

It may be verified directly that

$$\chi_l(x) = \chi_{l-1}(x) + ((x-\alpha)(x-\bar{\alpha}))^{l-1}(sx+t)$$

satisfies our requirement, where the real numbers s and t are given by

$$\lambda(\alpha) + 2(s\alpha + t)\chi_{l-1}(\alpha) = 0$$

(the existence of s and t is easily seen, since α is not real and $\chi_{l-1}(\alpha) \neq 0$).

3) Let $q_1(x)$ and $q_2(x)$ be two real polynomials without common divisor, and let $\chi_1(x)$ and $\chi_2(x)$ be two real polynomials satisfying

$$\chi_1^2(x) - x \equiv 0(\text{mod } q_1(x))$$

and

$$\chi_2^2(x) - x \equiv 0(\text{mod } q_2(x)).$$

It is well-known that we have two real polynomials $h_1(x)$ and $h_2(x)$ such that

$$h_1(x)q_1(x) + h_2(x)q_2(x) = 1.$$

Then on letting

$$\chi(x) = \chi_1(x)h_2(x)q_2(x) + \chi_2(x)h_1(x)q_1(x),$$

we have

$$\chi^2(x) - x \equiv 0(\text{mod } q_1(x)q_2(x)).$$

Applying the process repeatedly, we have the theorem.

7. A theorem on pairs of Hermitian forms

Theorem 15 *If H and K are two Hermitidn linear λ-matrices having the same elementary divisors, then we have two non-singular matrices Γ_1 and Γ_2 such that*③

$$\begin{aligned}\bar{\Gamma}_1 H \Gamma_1' &= h_1^{(r_1)} + h_2^{(r_2)}, \\ \bar{\Gamma}_2 K \Gamma_2' &= k_1^{(r_1)} + k_2^{(r_2)},\end{aligned} \quad r_1 + r_2 = h, \quad r_1 \geqslant 0, \quad r_2 \geqslant 0$$

③ In case $r_1 = 0$, $h(r_1)$ is left out.

and, we have two non-singular matrices $p_1^{(r_1)}$ *and* $p_2^{(r_2)}$ *such that*

$$\bar{p}_1 h_1 p_1' = k_1,$$
$$\bar{p}_2 h_2 p_2' = -k_2.$$

Proof 1) By the hypothesis we have two non-singular matrices P and Q such that

$$PHQ = K.$$

Since $PHQ = PH\bar{p}' \cdot \bar{p}'^{-1}Q$, we may assume that $P = I$.

Since H and K are both Hermitian we have

$$HQ = \bar{Q}'H = K.$$

We have a non-singular matrix T such that

$$T^{-1}QT = q_1^{(r_1)} \dot{+} q_2^{(r_2)},$$

where q_1 hsa non-negative characteristic roots and q_2 has only negative characteristic roots. We may assume without loss of generality that

$$Q = q_1 \dot{+} q_2.$$

Let

$$H = \begin{pmatrix} h_{11} & h_{12} \\ \bar{h}_{12}' & h_{22} \end{pmatrix}.$$

Since $HQ = \bar{Q}'H$, we have $h_{12}q_2 = \bar{q}_1' h_{12}$. Since $\bar{q}_1'$, q_2 have no common characteristic root, then $h_{12} = O$. Thus

$$H = h_1^{(r_1)} \dot{+} h_2^{(r_2)}.$$

Consequently

$$K = k_1^{(r_1)} \dot{+} k_2^{(r_2)}$$

and

$$h_1 q_1 = \bar{q}_1' h_1 = k_1, \quad h_2 q_2 = \bar{q}_2' h_2 = k_2.$$

2) In the lemma of 6 we take $q(x)$ to be the characteristic polynomial of q_1. Then we have a real polynomial $\chi(x)$ such that

$$\chi^2(q_1) = q_1.$$

Then, letting $p_1 = \chi(q_1)$, we have

$$k_1 = h_1 q_1 = h_1 \chi^2(q_1) = \chi(\bar{q}_1')h_1\chi(q_1) = \bar{p}_1' h_1 p_1.$$

Next, in the lemma of 6, we take $q(x)$ to be the characteristic polynomial of $-q_2$. Then we have a polynomial $\chi(x)$ such that

$$\chi^2(-q_2) = -q_2.$$

Let $p_2 = \chi(-q_2)$, then

$$k_2 = h_2 q_2 = -h_2\chi^2(-q_2) = -\chi(-\bar{q}_2')h_2\chi(-q_2) = -\bar{p}_2' h_2 p_2.$$

The theorem is then proved.

8. Canonical form of pairs of Hermitian forms

First of all, we introduce the following notations: Let

$$j^{(t)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be a t-rowed square matrix (a_{ij}) with

$$a_{ij} = \begin{cases} 1, & \text{for } i+j = n+1, \\ 0, & \text{otherwise} \end{cases}$$

and let

$$m^{(t)}(\lambda) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda \\ 0 & 0 & \cdots & \lambda & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \lambda & \cdots & 0 & 0 \\ \lambda & 1 & \cdots & 0 & 0 \end{pmatrix},$$

be a t-rowed square matrix (b_{ij}) with

$$b_{ij} = \begin{cases} \lambda, & \text{for } i+j = n+1, \\ 1, & \text{for } i+j = n+2, \\ 0, & \text{otherwise} \end{cases}$$

(in case $n = 1$, then $b_{11} = \lambda$).

Theorem 16 *Let (A, B) and (A_1, B_1) be two pairs of Hermitian matrices. Let* det $(\lambda A + B) = 0$ *have no real root and let A and A_1 be non-singular. A necessary and sufficient condition for the pairs to be conjunctive is that they have the same elementary dirisors. More definitely, given*

$$g_i = ((\lambda - \lambda_i)(\lambda - \bar{\lambda}_i))^{t_{i1}} \cdots ((\lambda - \lambda_k)(\lambda - \bar{\lambda}_k))^{t_{ik}},$$

where $1 \leqslant i \leqslant n$ and g_i divides g_{i+1} and $\sum t_{ij} = \frac{1}{2}n$. Let

$$J = \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix}$$

and

$$M = \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\lambda_i) & O \end{pmatrix},$$

where the $\sum$'s denote direct sums and. for $t_{ij} = 0$, the corresponding term is to be left out. Then $\lambda J - M$ has the preassigned g_i as its i-th elementary divisor. Further every pair of Hermitian matrices (A, B) with g_i as its i-th elementary divisor is conjunctive to (J, M).

Proof It is not difficult to verify that $\lambda J - M$ has g_i as its i-th elementary divisor.

1) In Theorem 15, we take

$$H = \lambda A - B, \quad K = \lambda J - M.$$

If $r_2 = 0$, the theorem is evident. If $r_1 = 0$, then we have a non-singular matrix P such that $\bar{P}HP' = -K$. Let

$$Q = \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} I^{(t_{ij})} & O \\ O & -I^{(t_{ij})} \end{pmatrix}.$$

Then $\bar{Q}JQ' = -J$ and $\bar{Q}MQ' = -M$. Thus $\bar{Q}\bar{P}HP'Q' = K$ and the theorem is true.

2) Consider first the particular case where we have

$$g_n = ((x - \alpha)(x - \bar{\alpha}))^{n/2}$$

and $g_{n-1} = \cdots = g_1 = 1$. $\lambda J - M$ cannot be conjunctive to a direct sum of two Hermitian matrices. For otherwise we would have two ono-singular matrices P and

Q such that

$$P(\lambda J - M)Q = \begin{pmatrix} p_1 & O \\ O & p_2 \end{pmatrix}$$

and p_1 and p_2 are Hermitian. Then either $(x-\alpha)^{n/2}$ or $(x-\bar{\alpha})^{n/2}$, and hence both, would divide det (p_1). This is impossible. Then we have either $r_1 = 0$ or $r_2 = 0$ in this case. The result is then true for this particular case.

3) If $r_1 \neq 0$, $r_2 \neq 0$, then we have to consider h_1 and h_2 in Theorem 15 separately. Applying induction on the number of the distinct invariant factors, we have the theorem.

Theorem 17 *Every pair* (A, B), det $(A) \neq 0$, *of Hermitian matrices is conjunctive to the following pair* (J, M), *where*

$$J = \dot{\sum_i}\dot{\sum_j} \epsilon_{ij} j^{(s_{ij})} \dot{+} \dot{\sum_i}\dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix},$$

$$M = \dot{\sum_i}\dot{\sum_j} \epsilon_{ij} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_i}\dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\bar{\lambda}_i) & O \end{pmatrix};$$

the first $\dot{\sum_i}$ *runs over all real roots of* det $(\lambda A + B) = 0$ *and the second* $\dot{\sum_i}$ *runs over all pairs of complex roots of* det $(\lambda A + B) = 0$, and $\varepsilon_{ij} = \pm 1$.

The proof of this theorem is completely analogous to that of Theorem 16.

Definition The pair of forms (J, M) obtained in Theorem 17 is called *the canonical form* of all the pairs conjunctive to it.

For a fixed c, we may arrange s_{ij} as

$$\begin{aligned} s_{i1} &= s_{i2} = \cdots = s_{i\alpha} > s_{i\alpha+1} = \cdots = s_{i\alpha+\beta} \\ &> s_{i\alpha+\beta+1} = \cdots = s_{i\alpha+\beta+\gamma} \\ &> \cdots = \cdots = s_{i\alpha+\beta+\cdots+\eta}. \end{aligned}$$

We set

$$\begin{aligned} \sigma_1^{(i)} &= \varepsilon_{i1} + \cdots + \varepsilon_{i\alpha} \\ \sigma_2^{(i)} &= \varepsilon_{i\alpha+1} + \cdots + \varepsilon_{i\alpha+\beta} \\ \sigma_3^{(i)} &= \varepsilon_{i\alpha+\beta+1} + \cdots + \varepsilon_{i\alpha+\beta+\gamma}. \end{aligned}$$

The constants $\sigma_1^{(i)}$, $\sigma_2^{(i)}$, $\cdots$ are called *the system of signatures of the pairs of forms with respect to the real root* c.

To each real root we have a system of signatures. The totality of all the elementary divisors and all the systems of signatures is called the system of elementary divisors with signatures.

9. Law of inertia

Theorem 18 *The system of elementary divisors with signatures characterize the conjunctivity of pairs of Hermitian matrices completely. More exactly, the elementary divisors and the systems of signatures are the same for all conjunctive pairs of Hermitian matrices (law of inertia); pairs with different elementary divisors or with the same elementary divisors but different systems of signatures are not conjunctive.*

Proof 1) It is known that if two pairs of Hermitian matrices are conjunctive, then their elementary divisors are the same. Further, it is evident that two canonical pairs with the same elementary divisors and the same system of signatures are conjunctive.

Thus it is sufficient to establish the result by showing that any two canonical pairs of Hermitian matrices with the same elementary divisors but different systems of signatures are not conjunctive.

2) Let (J, M) and

$$J_1 = \dot{\sum_i}\dot{\sum_j} \epsilon_{ij} j^{(s_{ij})} \dot{+} \dot{\sum_i}\dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix},$$

$$M_1 = \dot{\sum_i}\dot{\sum_j} \epsilon_{ij} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_i}\dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\bar{\lambda}_i) & O \end{pmatrix}$$

be two canonical pairs of Hermitian matrices with the same elementary divisors. If (J, M) and (J_1, M_1) are conjunctive, then we have a non-singular $n \times n$ matrix Γ such that

$$\bar{\Gamma}(J, M)\Gamma' = (J_1, M_1).$$

Then

$$\bar{\Gamma}(MJ^{-1}) = (MJ^{-1})\Gamma,$$

since $MJ^{-1} = M_1 J_1^{-1} = \bar{\Gamma}(MJ^{-1})\bar{\Gamma}^{-1}$. Since $J^2 = I$, and

$$MJ^{-1} = \dot{\sum_i}\dot{\sum_j} m^{s_{ij}}(c_i) j^{(s_{ij})} \dot{+} \dot{\sum_i}\dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) j^{(t_{ij})} \\ m^{(t_{ij})}(\bar{\lambda}_i) j^{(t_{ij})} & O \end{pmatrix},$$

we have

$$\Gamma = \dot{\sum_i} \Gamma_i \dot{+} \dot{\sum_j} \begin{pmatrix} \Gamma_{11}^{(i)} & \Gamma_{12}^{(i)} \\ \Gamma_{21}^{(i)} & \Gamma_{22}^{(i)} \end{pmatrix}$$

and

$$\bar{\Gamma}_i\left(\dot{\sum_j} m^{(s_{ij})}(c_i)j^{(s_{ij})}\right)=\left(\dot{\sum_j} m^{(s_{ij})}(c_i)j^{(s_{ij})}\right)\bar{\Gamma}_i.$$

Also

$$\bar{\Gamma}_i\left(\dot{\sum}\varepsilon_{ij}m^{(s_{ij})}\right)\Gamma_i'=\dot{\sum}\varepsilon_{ij}'j^{(s_{ij})},$$

$$\bar{\Gamma}_i\left(\dot{\sum}\varepsilon_{ij}m^{(s_{ij})}(c_i)\right)\Gamma_i'=\dot{\sum}\varepsilon_{ij}'m^{(s_{ij})}(c_i).$$

Thus it is sufficient to prove the theorem for the case with a unique real root c.

3) We require a

Lemma *Let H^A denote the adjoint matrix of H.*

(i) *If H and K are two conjunctive non-singular Hermitian λ-matrices, then H^A and K^A are conjunctive also; furthermore, if we arrange H^A and K^A as polynomials in λ, then their corresponding coefficients (which are matrices) are conjunctire.*

(ii) *If* $\det(H)\neq 0$ *and*

$$H=h_1\dot{+}h_2\dot{+}\cdots\dot{+}h_t,$$

then

$$\frac{H^A}{d(H)}=\frac{h_1^A}{d(h_1)}\dot{+}\frac{h_2^A}{d(h_2)}\dot{+}\cdots\dot{+}\frac{h_t^A}{d(h_t)}.$$

(iii)

$$(m^{(t)}(\lambda))^A=(-1)^{\frac{1}{2}(t-1)(t-2)}\begin{pmatrix}1 & -\lambda & \lambda^2 & \cdots & (-\lambda)^{t-1}\\ -\lambda & \lambda^2 & -\lambda^3 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ (-\lambda)^{t-1} & 0 & 0 & 0 & 0\end{pmatrix},$$

which is a t-rowed square matrix (a_{ij}) with

$$a_{ij}=\begin{cases}(-\lambda)^{i+j-2}, & \textit{for } i+j\leqslant t+1,\\ 0, & \textit{otherwise}.\end{cases}$$

All these results may be verified easily.

4) Since

$$\bar{\Gamma}(J,M))\Gamma'=(J_1,M_1),$$

we have

$$\bar{\Gamma}((\lambda-c)J+M)\Gamma'=(\lambda-c)J_1+M_1$$

for any λ. We write, dropping the subscript i,

$$M(\lambda)=(\lambda-c)J+M=\sum_j \varepsilon_j m^{(s_j)}(\lambda)$$

and

$$M_1(\lambda)=(\lambda-c)J_1+M_1=\dot{\sum_j} \varepsilon'_j m^{(s_j)}(\lambda).$$

They are conjunctive for any λ. Thus det $(M(\lambda))$ and det $(M_1(\lambda))$ have the same sign, i.e., $\prod_j \varepsilon_j^{s_j}=\dot{\sum_j}\varepsilon_j^{\prime(s_j)}$, since

$$\det\,(\varepsilon_j m^{(s_j)}(\lambda))=(-1)^{\frac{1}{2}s_j(s_j-1)}(\epsilon_j\lambda)^{s_j}.$$

Further, let

$$\prod \varepsilon_j^{s_j}(-1)^{\frac{1}{2}s_j(s_j-1)}=\epsilon,$$

$$\begin{aligned}M(\lambda)^A&=\varepsilon\lambda^n\left(\dot{\sum_j}\frac{\varepsilon j(m^{(s_j)}(\lambda))^A}{\det\,(m^{(s_j)}(\lambda))}\right)\\&=\varepsilon\dot{\sum_j}\varepsilon_j(-1)^{\frac{1}{2}s_j(s_j-1)}(m^{(s_j)}(\lambda))^A\lambda^{n-s_j}.\end{aligned}$$

The coefficient of λ^{n-s_1} is equal to

$$\epsilon(-1)^{(s_1-1)}\sum_{1\leqslant j\leqslant\alpha}\varepsilon_j\begin{pmatrix}1&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&0\end{pmatrix},$$

since $(-1)^{\frac{1}{2}s_j(s_j-1)}(-1)^{\frac{1}{2}s_j(s_j-2)}=(-1)^{(s_j-1)}$. By (i) of the lemma, the signature of this matrix is equal to that of the corresponding expression of $M_1(\lambda)$; hence

$$\sum_{j=1}^{\alpha}\varepsilon_j=\sum_{j=1}^{\alpha}\varepsilon'_j.$$

The coefficient of $\lambda^{n-s_{\alpha+1}}$ is of the form

$$\varepsilon\left(\dot{\sum_{1\leqslant j\leqslant\alpha}}\varepsilon_j P_j\right)\dot{+}\varepsilon(-1)^{(s_{\alpha+1}-1)}\dot{\sum_{\alpha+1\leqslant j\leqslant\alpha+\beta}}\varepsilon_j\begin{pmatrix}1&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&0\end{pmatrix}.$$

The corresponding expression of $M_1(\lambda)$ may be written as

$$\varepsilon\left(\dot{\sum_{1\leqslant j\leqslant\alpha}}\varepsilon_j P_j\right)\dot{+}\varepsilon(-1)^{(s_{\alpha+1}-1)}\dot{\sum_{\alpha+1\leqslant j\leqslant\alpha+\beta}}\varepsilon_j'\begin{pmatrix}1&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&0\end{pmatrix}$$

(by arranging the first part such that $\varepsilon_i=\varepsilon_i'$ for $1\leqslant i\leqslant\alpha$). Thus we have

$$\sum_{j=\alpha+1}^{\alpha+\beta}\varepsilon_j=\sum_{j=\alpha+1}^{\alpha+\beta}\varepsilon_j'.$$

The result follows by induction.

10. Normal form of hypercircles

Theorem 19 *Every hypercircle is conjunctive under $\mathfrak{G}$ to a hypercircle with the matrix*

$$\begin{pmatrix}H_1&O\\O&H_2\end{pmatrix},\quad H_1=\begin{pmatrix}h_1^{(r)}&O\\O&O\end{pmatrix},\quad H_2=\begin{pmatrix}h_2^{(r)}&O\\O&O\end{pmatrix},$$

where h_1 and h_2 may be expressed as two direct sums

$$h_1=\dot{\sum_i}\dot{\sum_j}\varepsilon_{ij}j^{(s_{ij})}\dot{+}\dot{\sum_i}\dot{\sum_j}\begin{pmatrix}O&j^{(t_{ij})}\\j^{(t_{ij})}&O\end{pmatrix}$$

and

$$h_2=\dot{\sum_i}\dot{\sum_j}\varepsilon_{ij}m^{(s_{ij})}(c_i)\dot{+}\dot{\sum_i}\dot{\sum_j}\begin{pmatrix}O&m^{(t_{ij})}(\lambda_i)\\m^{(t_{ij})}(\bar{\lambda}_i)&O\end{pmatrix},$$

where the c's are real and the λ's are complex numbers.

Proof By Theorem 14, we have only to consider the case with

$$H_1=\begin{pmatrix}h_1&O\\O&O\end{pmatrix},\quad H_2=\begin{pmatrix}h_2&O\\O&O\end{pmatrix},\quad \det(h_1)\neq 0.$$

Consider the pair of Hermitian matrices $(h_1^{-1}, \bar{h}_2)$.

By Theorem 17, we have a non-singuiar matrix γ such that

$$\bar{\gamma} h_1^{-1} \gamma' = \dot{\sum} \dot{\sum} \varepsilon_{ij} j^{(s_{ij})} \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix},$$

$$\bar{\gamma} \bar{h}_2 \gamma' = \dot{\sum} \dot{\sum} \varepsilon_{ij} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\bar{\lambda}_i) \\ m^{(t_{ij})}(\lambda_i) & O \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} \bar{\gamma}'^{-1} & O \\ O & I \end{pmatrix}, \quad D = \begin{pmatrix} \bar{\gamma} & O \\ O & I \end{pmatrix}, \quad B = C = O, \quad \mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

$\mathfrak{T}$ belongs to $\mathfrak{G}$. Then

$$\bar{\mathfrak{T}} \begin{pmatrix} \begin{pmatrix} h_1 & O \\ O & O \end{pmatrix} & \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} h_2 & O \\ O & O \end{pmatrix} \end{pmatrix} \mathfrak{T}'$$

gives the required form $\Bigg($notice that

$$\left(\dot{\sum_{i,j}} \dot{\sum} \varepsilon_{ij} j^{(s_{ij})} \dot{+} \dot{\sum_i} \dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix} \right)^2 = I \Bigg).$$

Theorem 20 *Every hypercircle with a matrix of the form given in Theorem 19 has a canonical discriminantal matrix. Apart from ϵ_{ij}, all other quantities in the expression of the matrix of the hypercircle are completely determined by its discriminantal matrix.*

The proof of the theorem needs only a direct verification.

Thus for a given discriminantal matrix we have only *a finite number* of hypercircles, more exactly, the number of hypercicles is $\leqslant 2n$. We have to consider further whether the forms given in Theorem 19 are equivalent. The answer will be given in 15.

11. Complete reducibility

Definition A sub-set $\mathfrak{C}$ of $\mathfrak{G}$ is said to be *completely reducible*, if we have a

transformation $\mathfrak{W}$ belonging to $\mathfrak{G}$ such that the elements of $\mathfrak{W}^{-1}\mathfrak{C}\mathfrak{W}$ are of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A=\begin{pmatrix} a_1 & O \\ O & a_2 \end{pmatrix}, \quad B=\begin{pmatrix} b_1 & O \\ O & b_2 \end{pmatrix}, \quad C=\begin{pmatrix} c_1 & O \\ O & c_2 \end{pmatrix}, \quad D=\begin{pmatrix} d_1 & O \\ O & d_2 \end{pmatrix}.$$

Theorem 21 *Let $\mathfrak{H}$ and $\mathfrak{K}$ be two hypercircles with the same discriminantal matrix $\mathfrak{D}$, and let* $\det(\mathfrak{D}-\lambda\mathfrak{F})=0$ *have more than one distinct root. The transformations which carry $\mathfrak{H}$ to $\mathfrak{K}$ are completely reducible. In particular, if $\mathfrak{H}=\mathfrak{K}$, they form a completely reducible group.*

Proof We may assume that

$$\mathfrak{D}=\begin{pmatrix} O & T \\ -T' & O \end{pmatrix},$$

where $T=t_1\dot{+}t_2$ and t_1 and t_2 have no common characteristic roots.

Suppose that $\overline{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'=\mathfrak{K}$ where $\mathfrak{T}$ belongs to $\mathfrak{G}'$, then $\mathfrak{T}\mathfrak{D}\mathfrak{T}'=\mathfrak{D}$. Since $\mathfrak{T}\mathfrak{F}\mathfrak{T}'=\mathfrak{F}$ we have $\mathfrak{T}'^{-1}=-\mathfrak{F}\mathfrak{T}\mathfrak{F}$. Then $\mathfrak{T}\mathfrak{D}=-\mathfrak{D}\mathfrak{F}\mathfrak{T}\mathfrak{F}$.

Put

$$\mathfrak{T}=\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} O & T \\ -T' & O \end{pmatrix}=\begin{pmatrix} O & T \\ -T' & O \end{pmatrix}\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$$

i.e.,

$$T'C=CT, \quad T'D=DT'$$
$$TA=AT, \quad TB=BT'.$$

Since t_1, t_2 have no common characteristic root, we have

$$A=a_1\dot{+}a_2, \quad B=b_1\dot{+}b_2,$$
$$C=c_1\dot{+}c_2, \quad D=d_1\dot{+}d_2.$$

The theorem follows.

In order to investigate the conjunctivity under $\mathfrak{G}$ of the forms in Theorem 19, we need only investigate the conjunctivity under $\mathfrak{G}$ of

$$h_1 = \dot{\sum} \varepsilon_i j^{(t_i)},$$

$$h_2 = \dot{\sum} \varepsilon_i m^{(t_i)}(c),$$

where c is a real number. The solution are quite different according to $c < 0, > 0$ or $= 0$.

12. Conjunctivity under $\mathfrak{G}$ for $c < 0$

Theorem 22 *The hypercircle with the matrix*

$$\begin{pmatrix} j^{(t)} & O \\ O & m^{(t)}(c) \end{pmatrix}$$

is conjunctive under $\mathfrak{G}$ *to that with*

$$-\begin{pmatrix} j^{(t)} & O \\ O & m^{(t)}(c) \end{pmatrix}$$

provided $c < 0$.

Proof We shall first establish the following preliminary result:

We have a real and symmetric matrix $s^{(t)}$ such that

$$sj^{(t)}s = -m(c),$$

if $c < 0$.

The result is true for $t = 1$, since

$$\sqrt{|c|} \cdot 1 \cdot \sqrt{|c|} = -c, \text{ i.e., } s = \sqrt{-c}.$$

The result is also true for $t = 2$, since

$$\begin{pmatrix} 0 & \sqrt{-c} \\ \sqrt{-c} & -\frac{1}{2}(\sqrt{-c})^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{-c} \\ \sqrt{-c} & -\frac{1}{2}(\sqrt{-c})^{-1} \end{pmatrix} = -\begin{pmatrix} 0 & c \\ c & 1 \end{pmatrix},$$

$$\text{i.e.,} \quad s = \begin{pmatrix} 0 & \sqrt{-c} \\ \sqrt{-c} & -\frac{1}{2}(\sqrt{-c})^{-1} \end{pmatrix}$$

Suppose that the theorem is true for t, then we shall prove that it is also true for $t + 2$, i.e., suppose we have s such that

$$sjs = -m(c)$$

and

$$\det\,(sj+\sqrt{-c}I^{(t)})\neq 0.$$

Then, we solve

$$\begin{pmatrix} 0 & 0 & z \\ 0 & s^{(t)} & w' \\ z & w & u \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & j^{(t)} & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & z \\ 0 & s & w' \\ z & w & u \end{pmatrix} = -m^{(t+2)}(c),$$

i.e., we find real numbers z, u and a t-dimensional vector w such that

$$z^2=-c,\quad w'z+sjw'=\begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\quad 2uz+wjw'=0.$$

The first equation gives $z=\sqrt{-c}$, the second is then soluble in w if and only if

$$\det\,(sj+\sqrt{-c}I^{(t)})\neq 0,$$

which is true by assumption, and from the third we then have the value of u. Set

$$s^{(t+2)}=\begin{pmatrix} 0 & 0 & z \\ 0 & s^{(t)} & w' \\ z & w & u \end{pmatrix},$$

where z, w, u are determined in this way; then $s^{(t+2)}$ satisfies

$$s^{(t+2)}j^{(t+2)}s^{(t+2)}=-m^{(t+2)}(c)$$

and

$$\det\,(s^{(t+2)}j^{(t+2)}+\sqrt{-c}I^{(t+2)})=-4c\det\,(s^{(t)}j^{(t)}+\sqrt{-c}I^{(t)})\neq 0.$$

The preliminary result is now proved. Let

$$\mathfrak{T}=\begin{pmatrix} O & s^{-1} \\ -s & O \end{pmatrix},$$

which belongs to $\mathfrak{G}$. Then

$$\mathfrak{T}\begin{pmatrix} j^{(t)} & O \\ O & m^{(t)}(c) \end{pmatrix}\mathfrak{T}'=\begin{pmatrix} s^{-1}m(c)s^{-1} & O \\ O & sjs \end{pmatrix}=-\begin{pmatrix} j^{(t)} & O \\ O & m(c) \end{pmatrix}.$$

The theorem follows.

Consequently, the sings ϵ_{ij} corresponding to a negative c_i in Theorem 19 may be replaeed by $+1$.

13. Conjunctivity under $\mathfrak{G}$ for $c > 0$

Theorem 23 *If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjunctive under $\mathfrak{G}$, then the two pairs of Hermitian matrices*

$$(\bar{\mathfrak{H}}_1, \mathfrak{F}\mathfrak{H}_1^A\mathfrak{F})$$

and

$$(\bar{\mathfrak{H}}_2, \mathfrak{F}\mathfrak{H}_2^A\mathfrak{F})$$

are also conjunctive under $\mathfrak{G}$.

Proof Let $\mathfrak{T}$ be an element of $\mathfrak{G}$ and $\bar{\mathfrak{T}}\mathfrak{H}_1\mathfrak{T}' = \mathfrak{H}_2$. Since $\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$ and $\mathfrak{T}^{-1} = \mathfrak{T}^A$ we have

$$\mathfrak{H}_2^A = \mathfrak{T}'^A\mathfrak{H}_1^A\mathfrak{T}^A = \mathfrak{T}'^{-1}\mathfrak{H}_1^A\mathfrak{T}^{-1}$$

and

$$\mathfrak{F}\mathfrak{H}_2^A\mathfrak{F} = \mathfrak{F}\mathfrak{T}'^{-1}\mathfrak{H}_1^A\mathfrak{T}^{-1}\mathfrak{F} = \mathfrak{T}\mathfrak{F}\mathfrak{H}_1^A\mathfrak{F}\mathfrak{T}'.$$

Therefore

$$\mathfrak{T}(\lambda\bar{\mathfrak{H}}_1 + \mu\mathfrak{F}\mathfrak{H}_1^A\mathfrak{F})\bar{\mathfrak{T}}' = \lambda\mathfrak{H}_2 + \mu\mathfrak{F}\mathfrak{H}_2^A\mathfrak{F}.$$

Theorem 24 *Let $c > 0$, and*

$$h_1 = \sum \varepsilon_i j^{(s_i)},$$

$$h_2 = \sum \varepsilon_i m^{(s_i)}(c).$$

For different systems of signatures we have non-conjunctive hypercircles (under $\mathfrak{G}$) with matrices

$$\mathfrak{H} = \begin{pmatrix} h_1 & O \\ O & h_2 \end{pmatrix}$$

under $\mathfrak{G}$.

Proof Let

$$\mathfrak{H} = \begin{pmatrix} h_1 & O \\ O & h_2 \end{pmatrix}, \quad \mathfrak{K} = \begin{pmatrix} k_1 & O \\ O & k_2 \end{pmatrix}$$

be two such hypercircles with different systems of signatures. If they are conjunctive under $\mathfrak{G}$, then

$$(\mathfrak{H}, -\mathfrak{F}\mathfrak{H}^A\mathfrak{F}) = \left(\begin{pmatrix} h_1 & O \\ O & h_2 \end{pmatrix}, \begin{pmatrix} \det (h_1)h_2^A & O \\ O & \det (h_2)h_1^A \end{pmatrix}\right)$$

and

$$(\mathfrak{K}, -\mathfrak{F}\mathfrak{K}^A\mathfrak{F}) = \left(\begin{pmatrix} k_1 & O \\ O & k_2 \end{pmatrix}, \begin{pmatrix} \det (k_1)k_2^A & O \\ O & \det (k_2)k_1^A \end{pmatrix}\right)$$

are conjunctive.

We shall now prove that

$$\phi = \lambda h_1 + \mu \det (h_1)h_2^A$$

is conjunctive to

$$\psi = \lambda h_2 + \mu \det (h_2)h_1^A.$$

We have

$$\begin{aligned} h_1^A \phi h_2 &= h_1^A(\lambda h_1 + \mu \det (h_1)h_2^A)h_2 \\ &= \det (h_1)(\lambda h_2 + \mu \det (h_2)h_1^A) = \det (h_1)\psi. \end{aligned}$$

Then

$$h_1^A \phi \bar{h}_1'^A \cdot h_1 h_2 = (\det (h_1))^2 \psi.$$

Now $[\det\ (h_1)]^2$ is positive and $h_1 h_2$ is a matrix wiht a positive characteristie root c. Hence as in the proof of Theorem 15, we have a matrix p such that

$$p h_1^A \phi \bar{h}_1'^A \bar{p}' = \psi.$$

Thus ϕ and ψ have the same system of elementary divisors with the same systems of signatures. Thus if $(\mathfrak{H}, \mathfrak{F}\mathfrak{H}^A\mathfrak{F})$ and $(\mathfrak{K}, \mathfrak{F}\mathfrak{K}^A\mathfrak{F})$ are conjunctive, then

$$(h_1, \det (h_2)h_1^A), \quad (k_2, \det (k_2)k_1^A)$$

are conjunctive, then (since $h_1^{-1} = h_1,\ k_1^{-1} = k_1$),

$$(h_2, h_1), \quad (k_2, k_1)$$

are conjunctive. By Theorem 16, they are conjunctive if and only if they have the same systems of signatures.

Consequently the signs ε_{ij} corresponding to a positive c_i in Theorem 19 are significant.

14. Conjunctivity under $\mathfrak{G}$ for $c = 0$

Here we require a preliminary lemma.

Lemma *Let*

$$t^{(l)} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

be an l-rowed matrix. The solution of

$$x^{(l,m)}t^{(m)} = t^{(l)}x^{(l,m)}$$

is of the form

$$x^{(l,m)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ 0 & x_1 & \cdots & x_{m-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{pmatrix} \quad if \quad l > m,$$

$$x^{(l,m)} = \begin{pmatrix} 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_l \\ 0 & \cdots & 0 & 0 & x_1 & \cdots & x_{l-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & x_l \end{pmatrix} \quad if \quad l < m,$$

$$x^{(l,l)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_l \\ 0 & x_1 & \cdots & x_{l-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_l \end{pmatrix}$$

Theorem 25 *Theorem 24 is also true for $c = 0$.*

Proof Let

$$\mathfrak{H} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \quad \mathfrak{K} = \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix}$$

and let

$$T = H_1'H_2 = K_1'K_2 = \sum j^{(s_i)}m^{(s_i)}(0),$$

where

$$j^{(s)}m^{(s)}(0)=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{pmatrix}$$

(for $s=1$, it is zero).

Evidently $H_1^2=K_1^2=I$. Let

$$\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}'=\mathfrak{K},\quad \mathfrak{T}=\begin{pmatrix}A&B\\C&D\end{pmatrix}.$$

Since $\mathfrak{T}\mathfrak{F}\mathfrak{T}'=\mathfrak{F}$, $\mathfrak{F}^2=-I^{(2n)}$ and $\mathfrak{T}\mathfrak{D}\mathfrak{T}'=\mathfrak{D}$, we have $\mathfrak{T}\mathfrak{D}\mathfrak{F}=\mathfrak{D}\mathfrak{F}\mathfrak{T}$. Now $\mathfrak{D}=\begin{pmatrix}O&T\\-T'&O\end{pmatrix}$; consequently, we have

$$AT=TA,\quad BT'=TB,\quad CT=T'C,\quad DT'=T'D.$$

Now we use Greek letters to denote matrices commutative with T. Then

$$A=\alpha,\quad B=\beta H_1,\quad C=H_1\gamma,\quad D=H_1\delta H_1,$$

since $T'=H_1TH_1$, Since

$$\overline{\begin{pmatrix}A&B\\C&D\end{pmatrix}}\begin{pmatrix}H_1&O\\O&H_2\end{pmatrix}\begin{pmatrix}A&B\\C&D\end{pmatrix}'=\begin{pmatrix}K_1&O\\O&K_2\end{pmatrix},$$

we have

$$K_1=\bar{A}H_1A'+\bar{B}H_2B'=\bar{\alpha}H_1\alpha'+\bar{\beta}H_1H_2H_1\beta'=\bar{\alpha}H_1\alpha'+T\bar{\beta}H_1\beta'.$$

Write

$$K_1=(k_{ij})_{1\leqslant i,j\leqslant k}$$

with

$$k_{ii}=\varepsilon'_{ij}j^{(s_i)},\quad k_{ij}=0\text{ for }i\neq j.$$

Similarly, we write

$$H_1=(h_{ij})_{1\leqslant i,j\leqslant k}$$

with

$$h_{ii} = \varepsilon_{ij} j^{(s_i)}, \quad h_{ij} = 0 \text{ for } i \neq j.$$

Further, we write

$$T = (t_{ij})$$

with

$$t_{ii} = j^{(s_i)} m^{(s_i)}(0), \quad t_{ij} = 0 \text{ for } i \neq j;$$

and finally, we write

$$\alpha = (a_{ij}), \quad a_{ij} = a_{ij}^{(s_i \cdot s_j)}.$$

Then

$$k_{ij} = \sum_{\lambda,\mu} \bar{a}_{i\lambda} h_{\lambda\mu} a'_{j\mu} + \sum_{\lambda\cdots} t_{i\lambda} \cdots .$$

Now we consider the element in the $(s_i, 1)$-position. The contribution from k_{ij} is either ε'_i for $i = j$ or 0 for $i \neq j$. The contribution from $\sum_{\lambda\cdots} t_{i\lambda} \cdots$ is zero, since the last row of $t_{i\lambda}$ is zero.

By the lemma, since

$$a_{ik} t_{kk} = t_{ii} a_{ik},$$

we have

$$a_{ik} = \begin{pmatrix} 0 & \cdots & 0 & r_{ik} & * & \cdots & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & x_{ik} \end{pmatrix} \quad \text{for } s_i > s_k,$$

$$\text{or} = \begin{pmatrix} x_{ik} & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_{ik} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{pmatrix} \quad \text{for } s_i < s_k,$$

$$\text{or} = \begin{pmatrix} x_{ik} & * & \cdots & * \\ 0 & x_{ik} & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_{ik} \end{pmatrix} \quad \text{for } s_i = s_k.$$

The element in the $(s_i, 1)$-position of $\bar{a}_{i\lambda} h_{\lambda\mu} a'_{j\mu}$ is zero for $\lambda \neq \mu$; is zero for $s_i < s_\lambda$; is zero for $s_j > s_\mu$; and is

$$\sum_{s\lambda = s_i = s_j} \bar{x}_{i\lambda} \varepsilon_\lambda x_{j\lambda} \quad \text{for} \quad s_\lambda = s_\mu = s_i = s_j.$$

Thus we obtain

$$\sum_{s\lambda = s_i = s_j} \bar{x}_{i\lambda} \varepsilon_\lambda x_{j\lambda} = \begin{cases} \epsilon'_i, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

Let all the elements s_v equal to s_μ be

$$s_{\eta+1}, \cdots, s_{\eta+\xi}.$$

Then

$$\begin{pmatrix} \varepsilon'_{\eta+1} & 0 & \cdots & 0 \\ 0 & \varepsilon'_{\eta+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varepsilon'_{\eta+\xi} \end{pmatrix} = \overline{(x_{ij})} \begin{pmatrix} \varepsilon_{\eta+1} & 0 & \cdots & 0 \\ 0 & \varepsilon_{\eta+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \varepsilon_{\eta+\xi} \end{pmatrix} (x_{ij})'.$$

Thus

$$\varepsilon_{\eta+1} + \cdots + \varepsilon_{\eta+\xi} = \varepsilon'_{\eta+1} + \cdots + \varepsilon'_{\eta+\xi}.$$

The result follows.

15. Canonical form of hypercircles

We now summarize the results of 10–14

Theorem 26 *Every hypercircle is conjunctive under $\mathfrak{G}$ to a hypercircle with the matrix*

$$\begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} h_1^{(r)} & O \\ O & O \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_2^{(r)} & O \\ O & O \end{pmatrix},$$

where h_1 and h_2 may be expressed as two direct sums

$$h_2 = \dot{\sum_{c_i \geqq 0}}\dot{\sum} \varepsilon_{ij} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_{c_i<0}}\dot{\sum} m^{(s_{ij})}(c_i) \dot{+} \dot{\sum_i}\dot{\sum_j} \begin{pmatrix} O & m^{(t_{ij})}(\lambda_i) \\ m^{(t_{ij})}(\lambda_i) & O \end{pmatrix}$$

and

$$h_1 = \dot{\sum_{c_i \geqq 0}}\dot{\sum} \cdot \varepsilon_{ij} j^{(s_{ij})} \dot{+} \dot{\sum_{c_i<0}}\dot{\sum} j^{(s_{ij})} \dot{+} \dot{\sum_i}\dot{\sum_j} \begin{pmatrix} O & j^{(t_{ij})} \\ j^{(t_{ij})} & O \end{pmatrix},$$

where the first double summation runs over non-negative c's, the second runs over negative c's and the third runs over all complex λ's.

Moreover, to each non-negative c, we may define the system of signatures as we did for the pairs of Hermitian matrices. Elementary divisors and systems of signatures characterize completely the conjunctivity of hypercircles under $\mathfrak{G}$.

Thus the problem of the conjunctivity of hypercircles under $\mathfrak{G}$ is now solved completely.

16. A final remark

The treatment is much simpler for the case of the group $\mathfrak{G}_{II}$ which consists of all transformations of the form

$$Z_1 = (AZ + B)(CZ + D)^{-1},$$

$$A\bar{B}' = B\bar{A}', \quad C\bar{D}' = D\bar{C}', \quad A\bar{D} - B\bar{C} = I.$$

It is evident that a transformation with the matrix

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $\mathfrak{G}_{II}$ if and only if

$$\bar{\mathfrak{T}}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}.$$

Correspondingly, the transformation of hypercircles may be written as

$$\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}' = \mathfrak{K}.$$

Thus, the pair $\mathfrak{H}$, $\mathfrak{K}$ are conjunctive under $\mathfrak{G}_{II}$ in the strict sense, if and only if the pairs of Hermitian matrices

$$(\mathfrak{H}, i\mathfrak{F}), \quad (\mathfrak{K}, i\mathfrak{F})$$

are conjunctive.

The classification of the hypereircles under $\mathfrak{G}_{II}$ is thus simply a straightforward application of the preceding results on pairs of Hermitian forms.

National Tsinghua Universiiy of China, Institute of Mathematics, Academia Sinica.

On the Theory of Fuchsian Functions of Several Variables *

1. Introduction

The paper contains a part of the author's general treatment of the theory of Fuchsian functions of several complex variables which may be considered as the first approximation of the author's precise results concerning Fuchsian functions of a matrix variable. The hypothesis is comparatively simple and weak. Broadly speaking any discontinuous group of automorphs of a bounded transitive space will fulfil our requirements for most purposes. In reviewing the history of the theory of automorphic functions of several variables, we find either that the group is too specific[2]① or that the hypotheses are too complicated[3]. The present treatment seems to be comparatively satisfactory in both respects.

The space is metrized in a way which seems to be simpler and more precise than the metric of S.Bergmann[4]. In §4, the author establishes that any discontinuous group of the space is properly discontinuous. The convergence of Poincaré theta series has been investigated and a criterion, which seems to be the best possible, has been obtained.

If we restrict ourselves to the case that the discontinuous group has a compact fundamental domain, we can go a good deal further. As an example, we give a generalization of Siegel's theorem concerning the dependence of automorphic forms.

As an illustration, the author gives a discussion for Picard's hyperabelian functions at the end of the paper.

2. Analytic (or pseudo-conformal) automorphs of a bounded domain

Let $\mathfrak{R}$ be a bounded domain of the $2n$-dimensional space

$$(z) = (z_1, \cdots, z_n), \quad z_k = x_k + iy_k, \quad 1 \leqslant k \leqslant n.$$

---

* Received August 20, 1945. Reprinted from the *Annals of Mathematics*, 1946, 47(2).

① A list of references is given at the end of the paper. The booklet of Behnke and Thullen[1] will be referred to as B-T in the paper.

If there is no ambiguity, we use simply z to denote the complex vector (z). The domain $\mathfrak{R}$ will be referred to as space $\mathfrak{R}$. Without lose of generality we may assume that the origin

$$(0) = (0, \cdots, 0)$$

is an interior point of the space $\mathfrak{R}$. Let Γ be the group of analytic automorphism of the space $\mathfrak{R}$. Suppose that Γ is transitive, i.e. that any interior point of $\mathfrak{R}$ may be carried to (0) by a transformation of the group Γ.

The analytic automorphism with the fixed point 0 form a subgroup Γ_0 of Γ. It is called the *group of stability*. It is known that the group Γ_0 is compact (H.Cartan[5]). More precisely, since every transformation of Γ_0 is uniquely determined by its linear terms, each transformation of Γ_0 may be expressed by a matrix which is formed by the coeffcients of linear terms. Moreover, owing to the compactness, by a linear transformation, we may write a transformation of Γ_0 as

$$w = t_{0,U}(z),$$

where U denotes a unitary matrix.

More definitely, let

$$U = (u_{ij}),$$

then the transformation

$$w = t_{0,U}(z)$$

denotes the analytic mapping

$$w_i = \sum_{j=1}^{n} u_{ij} z_j + \text{ terms of higher powers}$$

(Cf. B-T, Kapitel 6).

Consider the cosets of Γ/Γ_0. Let

$$w = t_{a,I}(z) \quad (I \text{ being identity})$$

be the transformation carrying (0) to $(a) = (a_1, \cdots, a_n)$. As a tends to zero, it approaches the identity transformation. Then the elements of Γ may be written as

$$w = t_{a,U}(z) = t_{a,I}(t_{0,U}(z)), \tag{1}$$

where a runs over all interior points of $\mathfrak{R}$ and U runs over all admissible unitary matrices in the group of stability. The transformation given by (1), with fixed (a),

form a coset of Γ/Γ_0, which carries (0) to (a). Such a formulation suggests that the geometry is not very far from the ordinary hyperbolic geometry.

It is known that a family of analytic functions bounded in the aggregate form a normal family. In particular

$$t_{a,U}(z)$$

form a normal family, that is, in any sequence of the mappings

$$t_{a_1,U_1}(z), \cdots, t_{a_n,U_n}(z), \cdots,$$

we can select a subsequence

$$t_{a_{n_i},U_{n_i}}(z)$$

such that each component approaches an analytic function as a limit.

Let

$$J_{a,U}(z)$$

be the functional determinant of the transformation

$$w = t_{a,U}(z).$$

Let b be a point on the boundary of the space then we have a sequence a_i such that

$$\lim_{a_i \to b} J_{a_i,U}(z) = 0.$$

In fact, by normality, we have a_i such that

$$t_{a_i,U}(z)$$

converges uniformly either to an automorph of $\mathfrak{R}$ or to a degenerate transformation. Since the limiting transformation carries 0 to b, it cannot be an automorph of $\mathfrak{R}$. Then we have the assertion (Cf. B-T, Satz 55).

3. Metrization of the space

We write

$$w = t_{a,U}^{-1}(z) = f(z, a, U),$$

as the inverse mapping of

$$z = t_{a,U}(w).$$

Then

$$f(a, a, U) = 0.$$

We have

$$dw_s = \sum_{t=1}^{n} dz_t a_{ts}, \quad a_{ts} = \frac{\partial w_s}{\partial z_t}.$$

Let $J_{a,U}(z)$ denote the matrix

$$(a_{ts}) = \left(\frac{\partial w_s}{\partial z_t}\right).$$

Theorem 1 *The Hermitian differential form*

$$(\overline{dz})\overline{J_{z,U}(z)}J_{z,U}(z)'(dz)'$$

is positive definite for z belonging to the space $\mathfrak{R}$ and is invariant under the group Γ, where $\bar{M}$ and M' denote the conjugate and the transposed matrices of M respectively.

Proof Since in $\mathfrak{R}$, the function

$$d(J_{z,U}(z)) \neq 0,$$

where $d(M)$ denote the determinant of the matrix M, the Hermitian form is evidently definite. Let

$$z = f(x, \beta, V)$$

and

$$w = f(f(x, \beta, V), \alpha, U) = f(x, \gamma, W). \tag{1}$$

Putting $x = \gamma$ in (1), we have

$$0 = f(f(\gamma, \beta, V), \alpha, U).$$

Consequently, we have

$$\alpha = f(\gamma, \beta, V). \tag{2}$$

We have

$$(dz) = (dx)J_{\beta,V}(x).$$

Further

$$\begin{aligned} J_{z,U}(z) &= (J_{\alpha,U}(z))_{\alpha=z} \\ &= (J_{\alpha,U}(f(x,\beta,V)))_{\alpha=f(x,\beta,V)} \\ &= (J_{\beta,V}(x))^{-1}\left(\frac{\partial f(x,\gamma,W)}{\partial x}\right)_{\gamma=x} \end{aligned}$$

by (2).

Further, from

$$t_{\gamma,W}(w) = t_{\gamma,U}(u),$$

we deduce that $u = 0$ implies $w = 0$, i.e.,

$$u = t_{0,X}(w),$$

where X is a unitary matrix. Then

$$f^{-1}(w,\gamma,W) = f^{-1}(f^{-1}(w,0,X),\gamma,U)$$

and consequently

$$f(f(w,\gamma,U),0,X) = f(w,\gamma,W).$$

We have immediately that

$$\left(\frac{\partial(f(w,\gamma,W))}{\partial w}\right)_{w=\gamma} = \left(\frac{\partial f(w,\gamma,U)}{\partial w}\right)_{w=\gamma}\left(\frac{\partial f(x,0,X)}{\partial x}\right)_{x=0}.$$

Thus, we obtain

$$\left(\frac{\partial f(x,\gamma,W)}{\partial x}\right)_{\gamma=x} = \left(\frac{\partial f(x,\gamma,U)}{\partial x}\right)_{\gamma=x} X$$

and

$$J_{z,U}(z) = (J_{\beta,U}(x))^{-1} J_{x,U}(x) X.$$

Then, we deduce

$$\begin{aligned}
&(\overline{dz})\overline{J_{z,U}(z)}J_{z,U}(z)'(dz)' \\
=&(\overline{dx})\overline{J_{\beta,V}(x)}\ \overline{J_{\beta V}(x)}^{-1}\overline{J_{x,U}(x)}\bar{X}X'(J_{x,U}(x))'(J_{\beta,V}(x))'^{-1}(J_{\beta,V}(x))'(dx)' \\
=&(\overline{dx})\overline{J_{x,U}(x)}J_{x,U}(x)'(dx)'.
\end{aligned}$$

The theorem is now established.

Incidentally, we have established also the following theorem:

Theorem 2 *The Hermitian differential form*

$$(\overline{dz})\overline{J_{z,U}(z)}J_{z,U}(z)'(dz)'$$

is independent of U.

Is the metric unique? We have the following answer.

Theorem 3 *If*

$$(\overline{dz})H(z,\bar z)(dz)', \quad H=(h_{ij}), \quad \bar H'=H$$

is a positive definite Hermitian form invariant under the group Γ, *then it is equivalent to the form given in Theorem* 1 *by a suitable choice of coordinate system. More precisely, we have a constant matrix* C *such that*

$$H(w,\bar w)=\overline{J_{z,U}(z)}J_{z\ U}(z)',$$

where $(w)=(z)C$.

Proof Let

$$z=f(w,\beta,V).$$

Then

$$\overline{J_{\beta,V}(w)}H(z,\bar z)J_{\beta,V}(w)'=H(w,\bar w).$$

In particular, for $z=0$, we have

$$H(\beta,\bar\beta)=\overline{J_{\beta,V}(\bar\beta)}H(0,0)J_{\beta,V}(\beta)'. \tag{1}$$

For $\beta=0$, we have

$$H(0,0)=\bar V H(0,0)V',$$

since

$$J_{0,V}(0)=V.$$

We may choose P such that

$$H(0,0)=\bar P P'.$$

Then

$$\bar P P'=\bar V\bar P P'V'.$$

Now $P^{-1}VP$ is unitary. Let

$$H(z,\bar z)=\bar P K(\bar z,\bar z)P'.$$

Then, from (1), we have

$$H(z,\bar z)=\overline{J_{z,V}(z)}\bar P P' J_{z,V}(z)',$$

i.e.,

$$K(z,\bar{z}) = \overline{P^{-1}J_{z,V}(z)P}(P^{-1}J_{z,V}(z)P)'.$$

The theorem follows.

Theorem 4 *If the group of stability is irreducible, then the nonsingular invariant differential form is unique up to a constant factor.*

With the notation of the proof of Theorem 3, we have

$$H(0,0) = \bar{V}H(0,0)V'.$$

The result follows easily.

Evidently we have

Theorem 5 *The volume element*

$$|d(J_{z,U}(z))|^2 dx_1 \cdots dx_n dy_1 \cdots dy_n$$

is invariant under the group Γ, *where* $d(M)$ *denotes the determinant of the matrix* M.

Remark The present result depends only on the property that the group of stability is compact. The boundedness of $\mathfrak{R}$ is not used in its full force. It seems to be true that the Riemannian curvature of the space $\mathfrak{R}$ is never positive. But the author failed to find a proof for this important theorem. Moreover the transitivity is also a non-essential assumption.

4. Discontinuous group

A subgroup G of the group Γ is called *discontinuous*, if every infinite sequence of the transformations of G does not converge to a transformation of Γ. G is called *properly discontinuous*, if the set of images of an inner point of $\mathfrak{R}$ has no limit point in $\mathfrak{R}$. Evidently every properly discontinuous group is discontinuous. Now we shall prove that

Theorem 6 *Every discontinuous group of the space* $\mathfrak{R}$ *is properly discontinuous.*

Proof Supposing the contrary, without loss of generality, we assume that

$$a_1, a_2, \cdots, a_n, \cdots$$

is a sequence of points equivalent to zero under the group G, and that the sequence approaches a. The corresponding transformations are

$$t_{a_1,U_1}(z), \cdots, t_{a_n,U_n}(z), \cdots.$$

They form a normal family, we have then a limit transformation

$$t_{a,U}(z),$$

where U is a limit element of the aggregate of unitary matrices $U_1, \cdots, U_n, \cdots$. This contradicts our supposition.

Theorem 7 *If* $w = t_{0,U}(z)$ *belongs to* G, *then* U *is of finite order. Consequently, there is a finite number of transformations of a discontinuous group having an interior fixed point.*

The theorem is evident, since an infinite sef of unitary matrices has always a limit element.

Definition A transformation with a fixed point interior to $\mathfrak{R}$ is called an *elliptic transformation.* The corresponding unitary matrix is called *multiplier* of the transformation.

Remark In the proof of Theorem 6, we do not need the full force on the boundedness of $\mathfrak{R}$. That the group of stability be compact will meet our requirement. The geometries having hypercircle as absolute with a regular metric given in Hua[6] satisfy the requirement, so that our Theorem 6 is still true for them.

5. Distance

Definition Let a and b be two points of the space. We define the distance $\Delta(a, b)$ between two points a and b to be the greatest lower bound of the absolute value of the integral

$$\int_C \sqrt{(\overline{dz})H(\bar{z}, z)(dz)'}$$

for all possible rectifiable curves C in the space $\mathfrak{R}$ connecting a and b.

We may easily establish the following properties:

1) For any pair of points a and b in $\mathfrak{R}$, $\Delta(a, b)$ is finite.

2) $\Delta(a, b) = \Delta(b, a)$.

3) $\Delta(a, b) = 0$ if and only if $a = b$.

In fact, surrounding a we construct a sphere S of radius ρ which separates a and b, and lies entirely in $\mathfrak{R}$. Let $\lambda(z)$ be the least characterisic root of $H(z, \bar{z})$ which is continuous in z, and let q be the least value which $\lambda(z)$ takes on the closure of S. Evidently $q > 0$. Then

$$\int_C (\overline{dz})H(dz)' \geqslant \int_C \lambda(q)(d\bar{z})(dz)',$$

where C' denotes the part of C lying in S. Then

$$\int_C (\overline{dz})H(dz)' \geqslant q\int_{C'}(\overline{dz})(dz)' \geqslant q\rho.$$

Then we have 3).

4) If a, b, c are three points of the space, we have

$$\Delta(a,c) \leqslant \Delta(a,b) + \Delta(b,c).$$

Consequently, $\Delta(a,b)$ is a continuous function of a and b.

5) Leg $w = t(z)$ be a transformation of Γ, we have

$$\Delta(a,b) = \Delta(t(a),t(b)).$$

6) As a tends to a boundary point c of $\mathfrak{R}$, we have

$$\lim_{a\to c}\Delta(a,b) = \infty.$$

In fact, we may assume $b = 0$. Let λ be the greatest lower bound of $\Delta(0,c)$ for c running over all boundary points of $\mathfrak{R}$. Given $\varepsilon > 0$, we have a point c and a curve C connecting c and 0 lying in $\mathfrak{R}$ except the terminal point c, such that

$$\int_C \sqrt{\overline{dz}H(dz)'} \leqslant \lambda + \varepsilon.$$

Taking a point f on C, we have

$$\int_C = \int_{(0,f)} + \int_{(f,c)} \leqslant \lambda + \varepsilon,$$

i.e.,

$$\Delta(c,f) \leqslant \lambda + \varepsilon - \Delta(0,f).$$

We have a transformation $w = t(z)$ of Γ carrying f into 0, and

$$\Delta(t(c),0) = \Delta(c,f) \leqslant \lambda + \varepsilon - \Delta(0,f).$$

Since $t(c)$ is also a boundary point, $\Delta(t(c),0) \geqslant \lambda$, and we have

$$\Delta(0,f) \leqslant \varepsilon$$

for any $\varepsilon > 0$. By 3) this is impossible.

7) The points x satisfying

$$\Delta(a,x) \leqslant \rho$$

form a compact set. It is called a non-Euclidean sphere with center a and radius ρ.

The topology defined in the Euclidean sense is equivalent to the topology defined by considering non-Euclidean spheres as a complete system of neighborhoods.

Remark If we can establish that the space $\mathfrak{R}$ is of non-negative Riemannian curvature, the distance between two points is equal to the length of the unique geodesic connecting both points. The theory will be more elegant than the present one.

6. Fundamental region

By the result of §4, G is enumerable. We may assume that 0 is not a fixed point. In fact, the fixed points of a transformation form a manifold of dimension $\leqslant 2n-1$ and there are an enumerable many of them, thus we may choose a point which is not a fixed point of all the transformations of G. By a transformation, we take it to be 0. Then we may omit the unitary matrix from the subscript, we arrange the transformations of the discontinuous group G in the following order

$$t_i(z) = t_{a_i,U_i}(z), \quad i = 0, 1, 2, \cdots,$$

according to

$$0 = \Delta(a_0, 0) < \Delta(a_1, 0) \leqslant \Delta(a_2, 0) \leqslant \cdots \leqslant \Delta(a_i, 0) \leqslant \cdots$$

and $t_0(z) = z$ is the identity. We have then

$$\lim_{i\to\infty} \Delta(a_i, 0) = \infty.$$

Definition A subdomain F of $\mathfrak{R}$ is called a *fundamental region* of a discontinuous group G, if the images of F under G covers $\mathfrak{R}$ without gaps and overlappings.

Theorem 8 *There exists a fundamental region. More definitely, let a be any point which is not a fixed point, the point z satisfying*

$$\Delta(z, a) \leqslant \Delta(z, t_i(a)), \quad \text{for } i = 1, 2, \cdots,$$

form a fundamental region which is denoted by $F(a)$. It is called the radial region with center a.

Proof In fact, to every point z, we have a nearest, in the non-Euclidean sense, point $t_k(a)$ in the aggregate $t_i(a) (i = 1, 2, \cdots)$, or one of its nearest points. Then it

belongs to $F(t_k(a))$ which is an image of $F(a)$. There is no point belonging to the interior of two $F(t_i(a))$, for otherwise

$$\Delta(z,t_k(a)) \leqslant \Delta(z,t_j(a)), \quad \text{for } j \neq k$$

and

$$\Delta(z,t_j(a)) \leqslant \Delta(z,t_k(a)).$$

Consequently

$$\Delta(z,t_j(a)) = \Delta(z,t_k(a)),$$

i.e. z is on the boundary of $F(t_j(a))$.

Theorem 9 *Every compact domain M in $\mathfrak{R}$ is covered by a finite number of images of $F(a)$ under G.*

Proof Let M_1 be the set of points obtained as the sum of a set of closed non-Euclidean spheres with center at any point m of M and radius $\Delta(m,a)$. Then M_1 is also compact. If the intersection of M and $F(t_k(a))$ is non-empty, then M_1 contains $t_k(a)$. In fact, let p be a point in the intersection, then

$$\Delta(p,t_k(a)) \leqslant \Delta(p,a).$$

By definition of $M_1, t_k(a)$ belongs to M_1. Since

$$\lim_{k\to\infty} \Delta(0,t_k(a)) = \infty,$$

we have the theorem.

Consequently, we have

Theorem 10 *If F(a) is compact, the compactness is independent of the choice of a.*

Proof Let $F(b)$ be the radial region with center b. By Theorem 11, $F(a)$ is covered by a finite number of $F(t_i(b))$'s. Let them be

$$F(t_{\lambda_1}(b)), \cdots, F(t_{\lambda_s}(b)).$$

Then $F(b)$ is covered by

$$F(t_{\lambda_1}^{-1}(a)), \cdots, F(t_{\lambda_s}^{-1}(a)).$$

In fact, let P be a point of $F(b)$, there is one of

$$t_{\lambda_1}(P), \cdots, t_{\lambda_s}(P)$$

lying in $F(a)$. Then P belongs to $F(t_{\lambda_i}^{-1}(a))$. Since $F(t_{\lambda_i}^{-1}(a))$ is compact, we have the theorem.

Definition *The bisecting manifold of two points a and b of $\mathfrak{R}$ is defined by the points x satisfying*

$$\Delta(x, a) = \Delta(x, b).$$

Thus the fundamental region is bounded by a number, finite or infinite, of bisecting manifolds.

Theorem 11 *If $F(a)$ is compact, it is bounded by a finite number of bisecting manifolds.*

Proof Let δ be the diameter of F, i.e. the least upper bound of the non-Euclidean distances between any two points of F. Suppose that the theorem is false. Let the bisecting manifolds be defined by

$$\Delta(a_\alpha, x) = \Delta(a, x), \quad \alpha = i_1, i_2, \cdots .$$

Then

$$\Delta(a_\alpha, a) \leqslant 2\delta.$$

The fact that

$$\lim_{\alpha\to\infty} \Delta(a_\alpha, a) = \infty$$

contradicts our previous inequality.

7. Lemmas

Lemma 1 *Let*

$$w_k = f_k(z_1, \cdots, z_n), \quad 1 \leqslant k \leqslant n \tag{1}$$

be an analytic mapping. Let

$$z_k = x_k + iy_k, \quad w_k = u_k + iv_k.$$

The mapping (1) induces a transformation of $2n$-dimensional real space. The Jacobian of the induced transformation is equal to the square of the absolute value of the Jacobian of (1).

Proof We have

$$dw_i = \sum_{j=1}^{n} dz_j \frac{\partial f_i}{\partial z_j}$$

of which the determinant is the Jacobian of (1). Taking conjugate complexes, we have

$$d\bar{w}_i = \sum_{j=1}^{n} \overline{dz_j}\,\overline{\frac{\partial f_i}{\partial z_j}}.$$

Thus

$$\frac{\partial(w_1, \cdots, w_n, \bar{w}_1, \cdots, \bar{w}_n)}{\partial(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)} = \left|\frac{\partial w}{\partial z}\right|^2.$$

Further

$$(w_k, \bar{w}_k) = (u_k, v_k)\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad (z_k, \bar{z}_k) = (x_k, y_k)\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

we have the result.

Lemma 2 *If $f(z_1, \cdots, z_n)$ is a function regular in and on the polycylinder C with center $(z_1^0, \cdots, z_n^0)$ and radius ρ,* i.e.

$$\left|z_i - z_i^0\right| \leqslant \rho, \quad 1 \leqslant i \leqslant n,$$

we have

$$\int_C \cdots \int |f(z_1, \cdots, z_n)|^2 dx_1 \cdots dx_n dy_1 \cdots dy_n \geqslant |f(z_1^0, \cdots, z_n^0)|^2 \pi^n \rho^{2n}.$$

The equality holds only when f is a constant.

Proof Let

$$f(z_1, \cdots, z_n) = \sum_{m_1, \cdots, m_n = 0}^{\infty} a_{m_1, \cdots, m_n}(z_1 - z_1^0)^{m_1} \cdots (z_n - z_n^0)^{m_n}.$$

The integral is equal to

$$\begin{aligned}
&\int_0^\rho \cdots \int_0^\rho \int_0^{2\pi} \cdots \int_0^{2\pi} \left|\sum a_{m_1, \cdots, m_n} \rho_1^{m_1} \cdots \rho_n^{m_n} e^{i(m_1\theta_1 + \cdots + m_n\theta_n)}\right|^2 \\
&\qquad \times \rho_1 \cdots \rho_n d\rho_1 \cdots d\rho_n d\theta_1 \cdots d\theta_n \\
=&(2\pi)^n \int_0^\rho \cdots \int_0^\rho \sum |a_{m_1, \cdots, m_n}|^2 \rho_1^{2m_1+1} \cdots \rho_n^{2m_n+1} d\rho_1 \cdots d\rho_n \\
\geqslant&(2\pi)^n \int_0^\rho \cdots \int_0^\rho |a_{0, \cdots, 0}|^2 \rho_1 \cdots \rho_n d\rho_1 \cdots d\rho_n \\
=&\pi^n \rho^{2n} |f(z_1^0, \cdots, z_n^0)|^2.
\end{aligned}$$

The last statement of the lemma follows by re-examining the inequality.

8. Poincaré theta series

We construct the theta series

$$\Theta_k(z) = \sum_{i=0}^{\infty} \left(\frac{\partial t_i(z)}{\partial z} \right)^k .$$

Theorem 12 *The theta series converges absolutely (and uniformly) for $k \geqslant 2\lambda$, if the integral*

$$\int_{\mathfrak{R}} d(H(z, \bar{z}))^{1-\lambda} dx_1 \cdots dx_n dy_1 \cdots dy_n$$

converges, where

$$H(z, \bar{z}) = \overline{J_{z,U}(z)} J_{z,U}(z)'.$$

Proof It is easy to see that we have a constant Ω such that

$$\Omega^{-1} \leqslant d(H(z, \bar{z})) \leqslant \Omega,$$

where $\Omega = \Omega(\mathfrak{R}^*) > 0$ for all z lying in a compact region $\mathfrak{R}^*$ in the interior of $\mathfrak{R}$.

We choose a polycylinder C with center z_0 and with radius ρ such that $t_i(C)$'s do not overlap. Then, by Lemma 2 of §5,

$$\begin{aligned} \pi^n \rho^{2n} \left| \frac{\partial(t_i(z))}{\partial z} \right|^{2\lambda}_{\text{at } z=z_0} &\leqslant \int_C \left| \frac{\partial t_i(z)}{\partial z} \right|^{2\lambda} dxdy \\ &\leqslant \Omega^{|1-\lambda|} \int_C \left| \frac{\partial t_i(z)}{\partial z} \right|^{2\lambda} d(H(z, \bar{z}))^{1-\lambda} dxdy \\ &= \Omega^{|1-\lambda|} \int_{t_i(C)} d(H(w, \bar{w}))^{1-\lambda} dudv. \end{aligned}$$

Thus we have

$$\pi^n \rho^{2n} \sum_{i=0}^{\infty} \left| \frac{\partial t_i(z)}{\partial z} \right|^{2\lambda}_{\text{at } z=z_0} \leqslant \Omega^{|1-\lambda|} \int_{\mathfrak{R}} d(H(w, \bar{w}))^{1-\lambda} dudv.$$

This establishes the absolute convergence of the series. Similarly, we have the uniformity of convergence.

Evidently the integral converges for $\lambda = 1$, we have the following

Theorem 13 *The theta series converges for $k \geqslant 2$.*

Is the constant given in Theorem 12 the best possible? The answer seems to be affirmative.

"Verzerrungssatz". Let $\mathfrak{R}^*$ be a compact subdomain of $\mathfrak{R}$. There exists a constant $\Lambda(>0)$ depending only on $\mathfrak{R}^*$ so that, for any two points z and z^* of $\mathfrak{R}^*$, we have

$$\Lambda^{-1} \leqslant J_{a,U}(z)/J_{a,U}(z^*) \leqslant \Lambda.$$

In general is this statement true? The author cannot answer it. Nevertheless for the four main types of symmetric bounded spaces the statement holds (Cf. the last section of the paper).

Theorem 14 *If the "Verzerrungssatz" is true and if the fundamental region is compact, the theta series diverges when the integral in Theorem 12 does.*

Proof Let F be the fundamental region. Then

$$\left|\frac{\partial t_i(z)}{\partial z}\right|^{2\lambda} \geqslant \Lambda_n^{-1} \int_F \left|\frac{\partial t_i(z)}{\partial z}\right|^{2\lambda} d(H(z,\bar{z}))^{1-\lambda} dxdy \quad \left(\int_F d(H(z,\bar{z}))^{1-\lambda} dxdy\right)^{-1}.$$

Let

$$\int_F d(H(z,\bar{z}))^{1-\lambda} dxdy = P,$$

which is finite. Then

$$\begin{aligned}\sum_{i=0}^{\infty} \left|\frac{\partial t_i(z)}{\partial z}\right|^{2\lambda} &\geqslant \Lambda^{-1} P^{-1} \sum_{i=0}^{\infty} \int_{t_i(F)} d(H(z,\bar{z}))^{1-\lambda} dxdy \\ &= \Lambda^{-1} P^{-1} \int_{\mathfrak{R}} d(H(z,\bar{z}))^{1-\lambda} dxdy.\end{aligned}$$

The theorem follows.

In case the fundamental region is compact we have many advantages. In fact the essential difficulty arising from the "parabloic vertices" disappears. Broadly speaking under this hypothesis the theory of automoprhic functions may be developed quite satisfactorily.

From the discussion of §3, we see that as we apply the transformation

$$w = t_{\alpha,V}(z),$$

we have

$$(dz) = (dw) J_{\alpha,V}(z)$$

and

$$J_{z,U}(z) = (J_{\alpha,V}(z))^{-1} J_{2,o}(z) W,$$

where W is a unitary matrix. By the method of adjugation, we may obtain invariant integrals of any dimension. By them, we may find some improvements of Theorem 12 for special groups.

9. Dependence of Fuchsian forms

In the present section we shall give a generalization of a theorem due to Siegel[7]

Definition An analytic function $f(z)$ of n complex variables $(z) = (z_1, \cdots, z_n)$ is called a *Fuchsian form of weight k with multiplier system v of a group G in the space* $\mathfrak{R}$, if it is meromorphic in the space $\mathfrak{R}$ and if

$$f(t_i(z)) = v(t_i)\left(\frac{\partial f_i(z)}{\partial z}\right)^k f(z)$$

for all transformations $t_i(z)$ belonging to G, where k is a constant and $v(t_i)$ is a number depending only of t_i.

The Fuchsian forms of weight 0 with multiplier system 1 of a group G are called the *Fuchsian functions* in the space $\mathfrak{R}$ of the group G.

By a well-known method (See, e.g. Picard of[2,3] or Blumenthal[4]), we may establish the following theorem.

Theorem 15 *For K large enough, there are $n+1$ algebraically (actually analytically) independent integral Fuchsian forms of dimension K with $v=1$. Consequently there are n algebraically (actually analytically) independent Fuchsian functions of the space $\mathfrak{R}$ with respect to the group G.*

In order to establish the dependence of $n+1$ Fuchsian functions, we use Siegel's quick method.

Let $L = L(G, k, v)$ denote the set of all integral Fuchsian forms of weight k and multiplier v. If f_1 and f_2 belong to L, so does $\lambda_1 f_1 + \lambda_2 f_2$, for any complex λ_1 and λ_2. Hence L is a vector space with certain dimension d (infinite or finite). Now we suppose that k is real and $v(t_i)$ is of absolute value 1.

Siegel found that d is finite for Fuchsian forms of a symmetric matrix-variable and for G having a compact fundamental domain. Now we shall extend the result to any circular region and then, by it establish the dependence of automorphic functions.

By the definition of circular region, we have, in Γ, the subgroup of transformations

$$(z_1, \cdots, z_n) = e^{i\theta}(w_1, \cdots, w_n).$$

Consequently

$$d(H(ze^{i\theta}, \bar{z}e^{i\theta})) = d(H(z, \bar{z})).$$

Thus, let λ be a scalar,

$$d(H(\lambda z), \bar{\lambda}\bar{z})$$

is a function of $\lambda\bar{\lambda}$. Let $\lambda\bar{\lambda} = t$; it is a function of t.

In order to prove the result, we need, besides the original idea of Siegel, the following self-evident lamma:

Lemma *For brevity, we write*

$$h(z) = h(z, \bar{z}) = d(H(z, \bar{z})).$$

Let

$$b = \max_{z \in F} \left(\frac{\sum_{i=1}^{n} \left(z_i \frac{\partial h}{\partial z_i} + \bar{z}_i \frac{\partial h}{\partial \bar{z}_i} \right)}{h(z)}; 1 \right),$$

where F denotes a compact region (later the fundamental region of the group G). Then, for $t > 0$, we have

$$\lim_{t \to 1} \frac{\log|h(t^{\frac{1}{2}} z)| - \log|h(z)|}{\log t} \leqslant b$$

for z belonging to F.

With this b instead of the b in Siegel's paper, we may establish that

Theorem 16 *Suppose that $\mathfrak{R}$ is a circular region and that the fundamental domain is compact. The dimension d of the vector space $L(G,k,v)$ is 0 for $k < 0$ and 0 or 1 for $k = 0$. It is finite and $\leqslant ck$ where $c = (n+1)\, b^n$.*

From Theorem 16, we may deduce, as Siegel that

Theorem 17 *Suppose that $\mathfrak{R}$ is a circular region, and that G possesses a compact fundamental domain. Fuchsian functions of the space $\mathfrak{R}$ with the group G form an algebraic functional field with exactly n independent elements.*

Remark The condition that $\mathfrak{R}$ is a circular region may be abolished, if we adopt the method due to Poincaré and Blumenthal[8], which is very much more elaborate than that due to Siegel. It is very likely that every bounded transitive space is a circular region. Thus the author presents such a treatment.

10. An introduction of vectorial automorphic functions

The preceding generalization of the theory of automorphic functions seems to be not the most appropriate one. It may be considered as an introduction of a way along which the author has proceeded to a certain extent. Such an idea will be proved to be fruitful in the study of automorphic functions of a matrix variable. The main idea may be described, in short, by the following phrase: "functions of vectorial variables with values over a vectorial domain".

Definition Let

$$f_1((z)), \cdots, f_n((z))$$

be n functions meromorphic of the n variables

$$(z) = (z_1, \cdots, z_n)$$

in $\mathfrak{R}$. For a transtormation

$$t_\lambda: \ w_i = t_i^{(\lambda)}(z_1, \cdots, z_n), \quad 1 \leqslant i \leqslant n$$

of G, we have

$$(f_1, \cdots, f_n)(w) = (f_1, \cdots, f_n)(z)(J(t_\lambda))^{-k},$$

where $J(t_\lambda)$ is the Jacobian matrix

$$\left(\frac{\partial w_i}{\partial z_j}\right).$$

Then $(f_1, \cdots, f_n)$ is called a *vectorial Fuchsian form of weight k.*

A vectorial Fuchsian form of weight 0 is called a *vectorial Fchsian function.* If the Jacobian of $f_1, \cdots, f_n$ is not identically zero, the vectorial form is said to be *non-degenerate.*

Theorem 18 *To each Fuchsian function we can construct a vectorial Fuchsian form of weight* -1.

Proof Let $\chi(z_1, \cdots, z_n)$ be a Fuchsian function. Then

$$\left(\frac{\partial_\chi}{\partial w_1}, \cdots, \frac{\partial_\chi}{\partial w_n}\right)(w) = \left(\frac{\partial_\chi}{\partial z_1}, \cdots, \frac{\partial_\chi}{\partial z_n}\right)(z)J(t_\lambda)$$

is a vectorial Fuchsian form of weight -1.

As a consequence of Theorem 15, we have

Theorem 19 *There exists a non-degenerate vectorial Fuchsian function.*

Theorem 20 *Suppose that G has a compact fundamental region A vectorial Fuchsian function takes each vector an equal number of times, except those values lying on manifolds of dimension*$<2n$.

The theorem is a consequence of Blumenthal's result[9] concerning the theory of eliminations.

From a vectorial Fuchsian function

$$(f_1, \cdots, f_n)(z),$$

we construct

$$\left(\frac{\partial f_i}{\partial z_j}\right) = J(z_1, \cdots, z_n).$$

Then

$$J(z_1, \cdots, z_n) = J(w_1, \cdots, w_n)J(t_\lambda),$$

which may be described as a "matrix Fuchsian form". Suppose that the vectorial function is nonsingular, we have

$$d(J(z_1, \cdots, z_n) = d(J(w_1, \cdots, w_n)d(J(t_\lambda)).$$

Thus to each non-degenerate vectorial Fuchsian function, we can construct a Fuchsian form of weight -1.

Let χ be any Fuchsian form of dimension k. Let $w_1, \cdots, w_n$ be a nondegenerate vectorial Fuchsian function. Then

$$\chi d(J(w_1, \cdots, w_n))^k = g$$

is a Fuchsian function. By Theorem 17, J may be expressed as a polynomial of $w_1, \cdots, w_n$, and w_{n+1}, where w_{n+1} is a properly chosen Fuchsian function. Then, we have

$$\chi(J(w_1, \cdots, w_n)^k = A(w_1, \cdots, w_n, w_{n+1})$$

From this formal footing, we may extend some of the classical results. Since the author finds no way to get rid of the condition that G possesses a compact fundamental domain, he will not proceed to give a full discussion of its development.

11. A digression

It was determined by E Cartan[10], that there are six types of irreducible bounded transitive symmctric spaces. Among them there are four general types and two special types. By means of the notation of matrices, the four general types may be expressed easily as follows.

(i) *Geometry of symmetric matrices.* Let Z denote n-rowed symmetric matrices. The space $\mathfrak{R}$ is formed by the points Z satisfying

$$I - Z\bar{Z} > 0$$

(for an Hermitian matrix H, $H > 0$ means that H is positive definite). The group of motion of the space $\mathfrak{R}$ is constituted by all transformations of the form

$$Z_1 = (AZ + B)(\bar{B}Z + \bar{A})^{-1},$$

where

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}' = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

This is known as Siegel's symplectic geometry.

(ii) *Geometry of skew symmetric matrices.* The space $\mathfrak{R}$ is formed by the points, defined by skew symmetric matrices Z, satisfying

$$I - \bar{Z}Z' > 0.$$

The group of motion of the space $\mathfrak{R}$ is constituted by all transformations of the form

$$Z_1 = (AZ + B)(-\bar{B}Z + \bar{A})^{-1},$$

where

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

(iii) *Geometry of rectangular matrices.* Let $Z = Z^{(m,n)}$ be an $m \times n$-rowed matrices. The space $\mathfrak{R}$ is formed by the points Z satisfying

$$I^{(m)} - \bar{Z}Z' > 0.$$

The group of motion ϕ is constituted by all transformations of the form

$$Z_1 = (AZ + B)(CZ + D)^{-1},$$

where

$$\begin{pmatrix} \overline{A^{(m)} \quad B^{(m,n)}} \\ C \quad\quad D \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 \\ 0 & -I^{(n)} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}' = \begin{pmatrix} I^{(m)} & 0 \\ 0 & -I^{(n)} \end{pmatrix}.$$

(iv) *Geometry of complex spheres* (given at §13).

The four types of geometries possess special non-Euclidean properties which enable us to improve our general results. The first three types are included in the discussion of the author's papers[6]. As an illustration of the present general treatment, we shall study the final case to a certain extent. Note that this is the case known as Picard's hyperabelian group.

12. A particular kind of geometry of matrices

The matrices of the present section are all real.

Let $X = X^{(2,n)}$ be a $2\times n$ rowed real matrix. The space $\mathfrak{R}_0$ is formed by the points X satisfying

$$I^{(2)} - XX' > 0. \tag{1}$$

The motion of the space is given by

$$X_1 = (AX + B)(CX + D)^{-1}, \tag{2}$$

where

$$\begin{pmatrix} A^{(2)} & B \\ C & D \end{pmatrix}\begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}' = \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix},$$

i.e.,

$$AA' - BB' = I^{(2)}, \quad AC' = BD', \quad CC' - DD' = -I^{(n)}, \tag{3}$$

and the determinant of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is equal to 1.

From (3), we have consequently

$$A'A - C'C = I^{(n)}, \quad A'B = C'D, \quad B'B - D'D = -I^{(2)} \tag{4}$$

and

$$X_1 = (XB' + A')^{-1}(XD' + C'). \tag{5}$$

Let dX denote the differential[2] of X. Then

$$\begin{aligned} dX_1 &= (XB' + A')^{-1}dXD' - (XB' + A')^{-1}dXB'(XB' + A')^{-1}(XD' + C') \\ &= (XB' + A')^{-1}dX(CX + D)^{-1}. \end{aligned} \tag{6}$$

Further,

$$\begin{aligned} I - X_1X_1' =&(XB' + A'^{-1})[(XB' + A')(BX' + A) \\ &- (XD' + C')(DX' + C)](BX' + A)^{-1} \end{aligned}$$

[2] Notice that $d(M)$ denotes the determinant value of M and dM denotes the differential of M.

$$=(XB' + A')^{-1}(I - XX')(BX' + A)^{-1} \tag{7}$$

and

$$I - X_1'X_1 = (CX + D)'^{-1}(I - X'X)(CX + D)^{-1}. \tag{7'}$$

Consequently

$$d(XB' + A')^2 = d(CX + D)^2,$$

since

$$d(I - X'X) = d(I - XX').$$

Then

$$d(XB' + A') = d(CX + D), \tag{8}$$

since

$$d\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1,$$

and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ form a continuous piece.

From (6),(7) and (7′), we obtain an invariant differential positive definite quadratic form

$$\sigma((I^{(2)} - XX')^{-1}dX(I^{(n)} - X'X)^{-1}dX'), \tag{9}$$

where $\sigma(M)$ denotes the trace of the matrix M.

The Jacobian of the transformation (2) is equal to

$$d(XB' + A')^{-2-n}$$

by (6). The volume element of the space is given by

$$d(I - XX')^{-(2+n)/2}\dot{X}, \tag{10}$$

where $\dot{X} = \pi dx_{rs}$ (here we use again the fact that

$$d(I - XX') = d(I - X'X)).$$

Given a point P of the space $\mathfrak{R}_0$. We have two matrices Q and R such that

$$Q(I - PP')Q' = I \tag{11}$$

and

$$R(I - P'P)R' = I \tag{12}$$

(notice that each of $I - PP' > 0$ and $I - P'P > 0$ implies the other)

Then

$$X_1 = Q(X - P)(-P'X + I)^{-1}R^{-1} \tag{13}$$

is a transformation of the space carrying P into zero. Thus the space $\mathfrak{R}_0$ is *transitive.*

The group of stability at 0 is evidently given by

$$X_1 = AXD^{-1}, \tag{14}$$

where A and D are orthogonal. Every transformation may be considered as a combination of (13) and (14).

The Jacobian of (13) is given by

$$\begin{aligned}&(d(XP' - I)d(Q))^{-2}d(-P'X + I)^{-n}d(R)^{-n}\\ =&(d(XP' - I))^{-2-n}(d(I - P'P))^{(2+n)/2},\end{aligned} \tag{15}$$

since $d(XP' - I^{(2)}) = d(P'X - I^{(n)})$.

Let $\mathfrak{R}^*$ be a compact region interior to $\mathfrak{R}_0$, to establish the "Verzerrungssatz," we have to prove that there exists a constant $\Omega(> 0)$ depending on $\mathfrak{R}^*$ such that

$$\Omega^{-1} \leqslant \left|\frac{d(X_1P' - I)}{d(X_2P' - I)}\right|^{-2-n} \leqslant \Omega, \tag{16}$$

for all X_1 and X_2 in $\mathfrak{R}^*$ and all P of $\mathfrak{R}_0$. This can be established easily by the argument of continuity. (Notice that (16) establishes the result for (13) and that for (14) it is evident.)

It may be verified by the method given previously (the first paper of Hua[6]) that the space possesses a non-positive Riemannian curvaature.

13. Geometry of complex spheres

The space $\mathfrak{R}$ is formed by complex vectors z satisfying

$$|zz'|^2 + 1 - 2\bar{z}z' > 0 \tag{1}$$

and

$$|zz'| < 1. \tag{2}$$

The group of motion of the space is constituted by

$$\begin{aligned}z_1 =&\left\{\left[\left(\frac{1}{2}(zz' + 1), \frac{i}{2}(zz' - 1)\right) A' + zB'\right]\begin{pmatrix}1\\ i\end{pmatrix}\right\}^{-1}\\ &\times\left\{\left(\frac{1}{2}(zz' + 1), \frac{i}{2}(zz' - 1)\right) C' + zD'\right\}.\end{aligned} \tag{3}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}' = \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix}$$

and A, B, C and D are real and

$$d\begin{pmatrix} A & B \\ C & D \end{pmatrix} = +1.$$

The relation between the present space $\mathfrak{R}$ and that the space $\mathfrak{R}_0$ given in §12 is given by the following one to one transformation

$$X = 2\begin{pmatrix} zz'+1 & i(zz'-1) \\ \overline{zz}'+1 & -i(\overline{zz}'-1) \end{pmatrix}^{-1}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \tag{4}$$

which is real.

In fact, the matrix $I - XX'$ is evidently conjunctive to

$$\overline{\begin{pmatrix} zz'+1 & i(zz-1) \\ \overline{zz}'+1 & -i(\overline{zz}'-1) \end{pmatrix}}\begin{pmatrix} zz'+1 & i(zz'-1) \\ \overline{zz}'+1 & -i(\overline{zz}'-1) \end{pmatrix}' - 4\overline{\begin{pmatrix} z \\ \bar{z} \end{pmatrix}}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}'$$
$$= 2\begin{pmatrix} 1+|zz'|^2 - 2\bar{z}z' & 0 \\ 0 & 1+|zz'|^2 - 2\bar{z}z' \end{pmatrix}.$$

Thus to each z of $\mathfrak{R}$, by (1) there is a point X of $\mathfrak{R}_0$.

Conversely, to each point $X = \begin{pmatrix} x \\ x_0 \end{pmatrix}$ of $\mathfrak{R}_0$, we have

$$\begin{aligned} 2z &= (zz'+1)x + i(zz'-1)x_0 \\ &= zz'(x+ix_0) + (x-ix_0). \end{aligned}$$

Consequently, we have

$$4zz' = (zz')^2(x+ix_0)(x+ix_0)' + 2(xx'+x_0x_0')zz' + (x-ix_0)(x-ix_0)', \tag{5}$$

which is a quadratic equation in zz'. Since

$$\left|\frac{(x+ix_0)(x+ix_0)'}{(x-ix_0)(x-ix_0)'}\right| = 1,$$

the equation (5) has a unique solution

$$zz' = \frac{2-(xx'+x_0x_0') - 2\sqrt{(1-xx')(1-x_0x_0')-(x_0x')^2}}{(x+ix_0)(x+ix_0)'}$$

with $|zz'| < 1$ for X belonging to $\mathfrak{R}_0$. Therefore the mapping is one to one.

Now we are going to find the Jacobian of the transformation (4). Let $w = x + ix_0$. Then (4) takes the form

$$2z = zz'w + \bar{w}. \tag{6}$$

We have

$$2\frac{\partial z_i}{\partial w_j} = 2\sum_{k=1}^{n}\left(z_k\frac{\partial z_k}{\partial w_j}\right)w_i + zz'\delta_{ij}, \tag{7}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. We write (7) in the form of matrices:

$$(I - P)\left(\frac{\partial z}{\partial w}\right) = \frac{1}{2}zz'I, \tag{8}$$

where $P = w'z$. Similarly

$$(I - P)\left(\frac{\partial z}{\partial \bar{w}}\right) = \frac{1}{2}I. \tag{9}$$

Combining (8) and (9) and their conjugate equations, we have

$$\begin{aligned}\left|\frac{\partial(z, \bar{z})}{\partial(w, \bar{w})}\right| &= 2^{-2n}|d(I - P)|^{-2}(1 - |zz'|^2)^n \\ &= 2^{-2n}|1 - wz'|^{-2}(1 - |zz'|^2)^n,\end{aligned} \tag{10}$$

since $d(I - P) = 1 - wz'$. From (6) and its conjugate equation, by eliminating $\bar{w}$, we have

$$2(\overline{zz'}z - \bar{z}) = (|zz'|^2 - 1)w$$

and

$$wz' = \frac{2(\bar{z}z - |zz'|^2)}{1 - |zz'|^2}.$$

Therefore, we have

$$\left|\frac{\partial(z, \bar{z})}{\partial(w, \bar{w})}\right| = \frac{(1 - |zz'|^2)^{n+2}}{2^{2n}(1 + |zz'|^2 - 2\bar{z}z')^2}. \tag{11}$$

From (3), we have

$$\rho z_1 = \left(\frac{1}{2}(zz' + 1), \frac{i}{2}(zz' - 1)\right)C' + zD',$$

$$\rho = \left(\left(\frac{1}{2}(zz' + 1), \frac{i}{2}(zz' - 1)\right)A' + zB'\right)\begin{pmatrix}1\\ i\end{pmatrix}.$$

We define λ_1 and λ_2 by

$$\rho(\lambda_1, \lambda_2) = \left(\frac{1}{2}(zz' + 1), \frac{i}{2}(zz' - 1)\right)A' + zB'.$$

We have

$$(\lambda_1, \lambda_2)\begin{pmatrix} 1 \\ i \end{pmatrix} = 1, \quad \text{i.e., } \lambda_1 + i\lambda_2 = 1. \tag{12}$$

Then

$$\begin{pmatrix} \rho & 0 \\ 0 & \bar{\rho} \end{pmatrix}\begin{pmatrix} \lambda_1, & \lambda_2; & z_1 \\ \bar{\lambda}_1, & \bar{\lambda}_2; & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(zz'+1), & \frac{i}{2}(zz'-1); & z \\ \frac{1}{2}(\overline{zz}'+1), & -\frac{i}{2}(\overline{zz}'-1); & \bar{z} \end{pmatrix}\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}.$$

Since

$$\overline{\begin{pmatrix} \frac{1}{2}(zz'+1), & \frac{i}{2}(zz'-1); & z \\ \frac{1}{2}(\overline{zz}'+1), & -\frac{i}{2}(\overline{zz}'-1); & \bar{z} \end{pmatrix}}\begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{1}{2}(zz'+1), & \frac{i}{2}(zz'-1); & z \\ \frac{1}{2}(\overline{zz}'+1), & -\frac{i}{2}(\overline{zz}'-1); & \bar{z} \end{pmatrix}' = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

we have

$$\overline{\begin{pmatrix} \lambda_1, & \lambda_2; & z_1 \\ \bar{\lambda}_1, & \bar{\lambda}_2; & \bar{z}_1 \end{pmatrix}}\begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix}\begin{pmatrix} \lambda_1, & \lambda_2; & z_1 \\ \bar{\lambda}_1, & \bar{\lambda}_2; & \bar{z}_1 \end{pmatrix}' = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

i.e.,

$$\lambda_1^2 + \lambda_2^2 - z_1 z_1' = 0. \tag{13}$$

Combining (12) and (13), we have

$$\lambda_1^2 - (1-\lambda_1)^2 - z_1 z_1' = 0,$$

i.e.,

$$\lambda_1 = \frac{1}{2}(1 + z_1 z_1').$$

Consequently

$$\lambda_2 = \frac{i}{2}(z_1 z_1' - 1).$$

We have, therefore,

$$\begin{pmatrix} \rho & 0 \\ 0 & \bar{\rho} \end{pmatrix}\begin{pmatrix} \frac{1}{2}(z_1z_1'+1), & \frac{i}{2}(z_1z_1'-1), & z_1 \\ \frac{1}{2}(\overline{z_1z}_1'+1), & -\frac{i}{2}(\overline{z_1z}_1'-1), & \bar{z}_1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}(zz'+1), & \frac{i}{2}(zz'-1); & z \\ \frac{1}{2}(\overline{zz}'+1), & -\frac{i}{2}(\overline{zz}'-1); & \bar{z} \end{pmatrix}\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}. \tag{14}$$

Consequently

$$X_1 = (A' + XB')^{-1}(C' + XD')$$
$$= (AX + B)(CX + D)^{-1}. \quad (15)$$

Now we have established the relationship between two kinds of geometries. This asserts again that the geometries of matrices play an essential role in the study of automorphic functions of several variables.

Translating the results of the last section, we have the following properties of the sqpace $\mathfrak{R}$:

1) The space is transitive.

2) The volume element is equal to

$$\frac{\dot{z}}{(1 + |zz'|^2 - 2\bar{z}z')^n}, \quad \dot{z} = \pi\, dx_r dy_r, \quad z = x + iy,$$

by (10) of §12 and (11) of §13. The integral of Theorem 12 converges for③ $\lambda > 1 - \frac{1}{n}$. In fact, for $\lambda \geqslant 1$, the assertion is evident, and, for $\lambda \leqslant 1$, we have

$$\int_{\mathfrak{R}} (1 + |zz'|^2 - 2\bar{z}z')^{-n(1-\lambda)}\dot{z} \leqslant \int_{1-\bar{z}z'>0} \frac{\dot{z}}{(1 - 2\bar{z}z')^{n(1-\lambda)}}, \quad (16)$$

which converges for $\lambda > 1 - \frac{1}{n}$, since $|zz'| < 1$ and $1 + |zz'|^2 - 2\bar{z}z' > 0$ imply $\bar{z}z' < 1$.

3) The "Verzerrungssatz" is true.

4) The space has a non-positive Riemannian curvature.

$$\int_{\mathfrak{R}_0} d(I - \bar{x}\bar{x}')^{\lambda}\dot{x} = \pi^{2n}2^n\Gamma(2\lambda + 1)/\Gamma(2\lambda + n + 2).$$

5) The group of stability at $\mathfrak{O}$ is given by the transformations with $B = C = 0$ and or thogonal A and D. In particular

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad D = I,$$

we have

$$z_1 = e^{-i\theta}z,$$

③ The constant seems not to be a best possible one. Correspondingly, for the space $\mathfrak{R}_0$, we can evaluate exactly the value

$$\int_{\mathfrak{R}_0} d(I - XX')^{\lambda}\dot{X} = \pi^{2n}2^n\Gamma(2\lambda + 1)/\Gamma(2\lambda + n + 2).$$

But for the present case the author is unable to evaluate the value of (16).

since

$$\left(\frac{1}{2}(zz'+1),\ \frac{i}{2}(zz'-1)\right)\left(\begin{array}{cc}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{array}\right)\left(\begin{array}{c}1\\ i\end{array}\right)=e^{i\theta},$$

the space is a circular region.

List of References

[1] Behnke U Thullen. *Theorie der Funktionen mehrerer komplexer Veränderlichen.* Ergebnisse der Math uihrer Grenzgebiete, Bd 3, Heft 3 Berlin: Julius Springer.

[2] Hyper-Fuchsian group. Picard. *Acta Math*, 1882, 1:297; 1883, 2:114; 1884, 5:121. Cf also. Picard. *Jour de Math*, 1885, 1(4):357. *Bull. de la Soc. Math de France*, 1887, 15:148; *Alezias, Ann. scient Ec. Norm Sup*, 1902, 19(3):201 (also Thèse, Paris, 1901) and Craig, *Trans. Amer Math Soc.*, 1910, 11:37.

[3] Fubini, see 3 of [2].
Wirtinger, Wien. Berichte, 1899, 108:1239.
Girand, Lecons sur les fonctions automorphes, Paris: Gauthier-Villars, 1920.

[4] Bergmann. *Jour. für. Math.*, 1933, 169.

[5] H Cartan. *Jour. de Math.*, 1931, 10(9):1.

[6] Hua. *Amer. Jour. of Math.*, 1944, 66:470–488, 531–563.

[7] Siegel. *Annals of Math.*, 1942, 43(2):613–616.

[8] Poincaré. *Acta Math.*, 1902, 26:48–98. Blumenthal, see 4 of [2].

[9] Blumenthal. *Math. Annalen*, 1903, 57.

[10] E. Cartan *Abh. Math. Sem.* Hamburg Univ., 1935, 11:116–162.
(Note that some of the references on the list are not available in China. The author found these titles through indirect sources.)

TSING HUA UNIVERSITY KUNMING, CHINA

On the Extended Space of Several Complex Variables (I): the Space of Complex Spheres*

In the theory of functions of a complex variable we introduce the point at infinity to make the extended space(Cauchy plane)compact. The procedure of introducing point at infinity depends, in effect, on the group G of linear fractional transformations

$$z^* = (az+b)/(cz+d), \tag{1}$$

where $ad - bc \neq 0$. It is well known that the group G is "complete" in the sense that any analytic automorph of the extended space is a transformation of the from (1). The aim of the present series of papers is to extend this discussion to the study of functions of several complex variables.

Let G be a group of transformations

$$z_i' = f_i(z_1, \cdots, z_n) \tag{2}$$

of n complex variables. Suppose that G satisfies the assumptions given by Osgood①. The manifold at infinity is introduced by means of the group (2). The totality of finite points and the points at infinity form the extended space $\mathfrak{R} = \mathfrak{R}(G)$. Immediately, we have the problem: "is the group G which defines the extended space $\mathfrak{R}(G)$ complets?" More precisely, we seek the group G such that $\mathfrak{R}(G)$ admits no analytic automorph other than those of G.

This importhant problem has been answered, so far as I am aware, only for two very special cases: (i) the space of the theory of functions, in which the group G is given by

$$z_i^* = (\alpha_k z_k + \beta_k)/(\gamma_k z_k + \delta_k) \quad (\alpha_k\delta_k - \beta_k\gamma_k \neq 0; 1 \leqslant i, k \leqslant n), \tag{3}$$

and (ii) the extended projective space, in which the group is given by

$$z_i^* = (c_1^{(i)} z_1 + \cdots + c_n^{(i)} z_n + c_0^{(i)})/(c_1^{(0)} z_1 + \cdots + c_n^{(0)} z_n + c_0^{(0)}), \tag{4}$$

* Received 1 October 1945. Reprinted from the Quarterly Journal of Mathematics, Oxford series, 1946, 17(68): 214–222.

① *Lehrbuch der Funktionentheorie*, II_1. Teubner, 1929: 293 301.

where $(c_i^j)_{0\leqslant i,j\leqslant n}$ is a non-singular matrix.

Recently I established results for several other groups, and they seem to form a complete system in the sense of the structure of groups. The present paper contains a proof for the space of complex spheres. I hope to give the other cases in succeeding papers.

In the Lie geometry of hyperspheres in the $(n-1)$-dimensional space, we introduce 'homogeneous coordinates' $(u_1,\cdots,u_n,v_1,v_2)$ to represent a hypersphere with centre $(\xi_1,\cdots,\xi_{n-1})$ and radius R by means of the relations:

$$u_1^2+\cdots+u_n^2-v_1^2-v_2^2=0, \tag{5}$$

$$\left.\begin{aligned} &u_i=\rho\xi_i \quad (1\leqslant i\leqslant n-1),\\ &v_2=\rho R,\\ &u_n=\frac{1}{2}\rho\left(1-\sum_{i=1}^{n-1}\xi_i^2+R^2\right),\\ &v_1=\frac{1}{2}\rho\left(1+\sum_{i=1}^{n-1}\xi_i^2-R^2\right), \end{aligned}\right\} \tag{6}$$

and, inversely, we have

$$\xi_i=\frac{u_i}{u_n+v_1}, \qquad R=\frac{v_2}{u_n+v_1}.$$

The elements with $u_n+v_1=0$ represent improper hyperspheres. The Lie group of the geometry is given by

$$(u_1^*,\cdots,u_n^*,v_1^*,v_2^*)=\rho(u_1,\cdots,u_n,v_1,v_2)F, \tag{7}$$

where F is an $(n+2)$-rowed matrix leaving the quadratic relation (5) invariant. In non-homogeneous coordinates we have

$$\left.\begin{aligned} &\xi_i^*=f_i(\xi_1,\cdots,\xi_{n-1},R) \quad (1\leqslant i\leqslant n-1),\\ &R^*=f_n(\xi_1,\cdots,\xi_{n-1},R). \end{aligned}\right\} \tag{8}$$

If now we extend the geometry to the complex field, we obtain an extended space $\mathfrak{R}$ defined by n complex variables $(\xi_1,\cdots,\xi_{n-1},R)$, and the new group is obtained from (8) by varying F in the complex field and preserving the relation(5).

For the sake of convenience in the complex filed, we can modify our notation slightly. Write

$$\xi_1=z_1,\cdots,\xi_{n-1}=z_{n-1}, iR=z_n. \tag{9}$$

Now the homogeneous coordinates of the 'complex sphere' $(z_1, \cdots, z_n)$ are given by

$$x_i = \rho z_i, \quad y_i = \rho \sum_{i=1}^{n} z_i^2, \quad y_2 = \rho. \tag{10}$$

The transformation takes the form

$$(x_1^*, \cdots, x_n^*, y_1^*, y_2^*) = \rho(x_1, \cdots, x_n, y_1, y_2)F, \tag{11}$$

where F leaves the quadratic relation

$$\sum_{i=1}^{n} x_i^2 - y_1 y_2 = 0 \tag{12}$$

invariant. Write F as

$$\begin{pmatrix} T & v_1' & v_2' \\ u_1 & a & b \\ u_2 & c & d \end{pmatrix}, \tag{13}$$

where T is an n-rowed matrix and u_1, u_2, v_1, v_2 denote four n-vectors. Now corresponding to (8), we have

$$(z_1^*, \cdots, z_n^*) = \frac{(z_1, \cdots, z_n)T + u_1 \sum_{i=1}^{n} z_i^2 + u_2}{(z_1, \cdots, z_n)v_2' + b \sum_{i=1}^{n} z_i^2 + d} \tag{14}$$

Using (14) instead of (2), I shall prove here that *the group G defined by* (14) *contains all analytic automorphs of the space $\mathfrak{R}(G)$*.

We begin by finding the Laguerre subgroup H of the Lie group G: that is the group of transformations carrying improper points into improper points. More definitely, we look for those(14)which have no variables in the denominator. Consequently $v_2 = 0$, $b = 0$. From

$$\begin{pmatrix} T & v_1' & 0 \\ u_1 & a & 0 \\ u_2 & c & d \end{pmatrix} \begin{pmatrix} I^{(n)} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} T' & u_1' & u_2' \\ v_1 & a & c \\ 0 & 0 & d \end{pmatrix} = \rho^2 \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \tag{15}$$

we deduce that

$$TT' = \rho^2 I, \qquad u_1 = 0,$$

$$v_1 = \frac{2}{d} u_2 T', \qquad ad = \rho^2, \qquad c = \frac{u_2 u_2'}{d}.$$

Therefore

$$F = \begin{pmatrix} \rho\Gamma & 2\rho\Gamma u_2'/d & 0 \\ 0 & \rho^2/d & 0 \\ u_2 & u_2 u_2'/d & d \end{pmatrix}, \tag{16}$$

where Γ is orthogonal, F is a product of

$$\begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} \Gamma & 2u_2\Gamma & 0 \\ 0 & 1 & 0 \\ u_2 & u_2 u_2' & 1 \end{pmatrix} \tag{18}$$

and

$$\begin{pmatrix} I & 0 & 0 \\ 0 & 1/d & 0 \\ 0 & 0 & d \end{pmatrix}. \tag{19}$$

Corresponding to (18), we have the mapping

$$\begin{aligned}(z_1^*, \cdots, z_n^*) &= (x_1^*, \cdots, x_n^*)/y_2^* \\ &= \{(x_1, \cdots, x_n)\Gamma + y_2 v_2\}/y_2 = (z_1, \cdots, z_n)\Gamma + u_2.\end{aligned} \tag{20}$$

Corresponding to (17)and (19), we have the mapping

$$(z_1^*, \cdots, z_2^*) = \rho(z_1, \cdots, z_n). \tag{21}$$

The group H is generated by (20)and(21).

It follows that any two finite points are equivalent under the transformation (20). For points (i.e. improper complex spheres) at infinity, I make the assertion:

Every point at infinity is carried into a finite point by means of one of the following $n+1$ transformations:

$$z_i^* = -z_i \bigg/ \left(\sum_{j=1}^{n} z_j^2 \right) \qquad (1 \leqslant i \leqslant n), \tag{22}$$

and, for a fixed $p(1 \leqslant p \leqslant n)$,

$$\left.\begin{aligned} z_i^* &= z_i \Big/ \left(1 - 2z_p + \sum_{j=1}^n z_j^2\right) \quad (i \neq p), \\ z_p^* &= \left(-z_p + \sum_{j=1}^n z_j^2\right) \Big/ \left(1 - 2z_p + \sum_{j=1}^n z_j^2\right). \end{aligned}\right\} \tag{23}$$

In fact, write

$$b = (x_1, \cdots, x_n, y_1, y_2).$$

Let a be a vector $(a_1, \cdots, a_n, b_1, b_2)$. Then, evidently

$$b^* = -\frac{2\langle a, b\rangle}{\langle a, a\rangle} a + b \tag{24}$$

is a transformation of the extended space, where

$$\langle a, b\rangle = \sum_{j=1}^n a_j x_j - \frac{1}{2}(b_1 y_2 + b_2 y_1) \tag{25}$$

(actually, this is known as Lie inversion in the geometry of spheres). Suppose that b is a point at infinity: that is, it has $y_2 = 0$. Then we have

$$y_2^* = -\frac{2\langle a, b\rangle}{\langle a, a\rangle} b_2. \tag{26}$$

Evidently, at least one of the vectors

$$a = (0, \cdots, 0, 0, 1, 1) \tag{27}$$

and

$$a = (0, \cdots, \underset{p\text{-th}}{1}, 0, \cdots, 0, 1) \tag{28}$$

makes (26) non-vanishing.

Corresponding to (27) we have the transformation

$$z_i^* = \frac{x_i^*}{y_2^*} = \frac{x_i}{-y_1} = -\frac{z_i}{\sum\limits_{j=1}^n z_j^2}. \tag{29}$$

Corresponding to (28) we have

$$b^* = -(2x_p - y_1)(0, \cdots, 1, 0, \cdots, 0, 1) + b,$$

i.e.,

$$x_i^* = x_i \quad (i \neq p), \qquad x_p^* = -(2x_p - y_1) + x_p = -x_p + y_1,$$

$$y_1^* = y_1, \qquad y_2^* = -(2x_p - y_1) + y_2.$$

Then, when $i \neq p$,

$$z_i^* = \frac{x_i^*}{y_2^*} = \frac{x_i}{-2x_p + y_1 + y_2} = \frac{z_i}{1 - 2z_p + \sum_{j=1}^{n} z_j^2} \tag{30$_1$}$$

and

$$z_p^* = x_p^*/y_2^* = (-x_p + y_1)/(-2x_p + y_1 + y_2) = \frac{-z_p + \sum_{j=1}^{n} z_j^2}{1 - 2z_p + \sum_{j=1}^{n} z_j^2}. \tag{30$_2$}$$

Notice that (30) may be obtained from (29) by means of the transformations

$$z_i = w_i \quad (i \neq p), \qquad z_p = 1 - w_p,$$

and

$$z_i^* = -w_i^* \quad (i \neq p), \qquad z_p^* = -(1 - w_p^*).$$

The Jacobian of the transformation (29) is equal to

$$\left(\sum_{j=1}^{n} z_j^2\right)^{-n} \tag{31}$$

and that of (30) is equal to

$$\left(1 - 2z_p + \sum_{j=1}^{n} z_j^2\right)^{-n}. \tag{32}$$

I now assume that $n \geqslant 3$. Let

$$z_i^* = f_i(z_1, \cdots, z_n) \quad (1 \leqslant i \leqslant n) \tag{33}$$

be an analytic mapping carrying the extended space of complex spheres into itself. By a theorem due to Osgood, † the mapping (33) is birational. Consequently, (33) can be written as

$$z_i^* = p_i(z_1, \cdots, z_n)/q(z_1, \cdots, z_n), \tag{34}$$

where $p_i (1 \leqslant i \leqslant n)$ and q are $n+1$ polynomials without common divisor other than constant.

1. There is a point $(z_1^{(0)}, \cdots, z_n^{(0)})$ satisfying

$$p_i(z_1^{(0)}, \cdots, z_n^{(0)}) \neq 0$$

and

$$q(z_1^{(0)}, \cdots, z_n^{(0)}) \neq 0.$$

The transformation

$$z_i = w_i + z_i^{(0)},$$

of the form (20), converts (34) into a new transformation in which

$$p_i(0, \cdots, 0) \neq 0, \qquad q(0, \cdots, 0) \neq 0. \tag{35}$$

The transformation

$$z_i^* = f_i \left(\frac{-z_1}{\sum_j z_j^2}, \cdots, \frac{-z_n}{\sum_j z_j^2} \right), \tag{36}$$

which is the product of (33) and (22), also maps the extended space on itself. Write

$$z_i^* = \left\{ p_i \left(\frac{-z_1}{\sum z_j^2}, \cdots, \frac{-z_n}{\sum z_j^2} \right) \left(\sum z_j^2 \right)^\lambda \right\} \Big/ \left\{ q \left(\frac{-z_1}{\sum z_j^2}, \cdots, \frac{-z_n}{\sum z_j^2} \right) \left(\sum z_j^2 \right)^\lambda \right\},$$

where λ is the least integer that makes all the numerators and the denominator integral. On account of (35), we find that

$$p_i \left(\frac{-z_1}{\sum z_j^2}, \cdots, \frac{-z_n}{\sum z_j^2} \right) \left(\sum z_j^2 \right)^\lambda$$

and

$$q \left(\frac{-z_1}{\sum z_j^2}, \cdots, \frac{-z_n}{\sum z_j^2} \right) \left(\sum z_j^2 \right)^\lambda$$

are all of degree 2λ.

Consider the Jacobian of (36). Let Δ and Δ_1 be the respective inverses of the Jacobians of (33) and (36). Δ and Δ_1 are polynomials; for, otherwise, there would exist points making the Jacobian vanish. From (36) and (31) we have

$$\Delta_1(z_1, \cdots, z_n) = \Delta\left(\frac{-z_1}{\sum z_j^2}, \cdots, \frac{-z_n}{\sum z_j^2}\right)\left(\sum z_j^2\right)^n. \tag{37}$$

Since $q(0, \cdots, 0) \neq 0$, we have $\Delta(0, \cdots, 0) \neq 0$. Consequently, $\Delta_1(z_1, \cdots, z_n)$ is a polynomial of degree $2n$.

Now we may assume, without loss of generality, that p_j and q are polynomials of degree 2λ, that their terms of the highest degree are constant multiples of $\left(\sum z_j^2\right)^\lambda$, and that the Jacobian Δ of (34) is a polynomial of degree $2n$ and its terms of highest degree a constant multiple of $\left(\sum z_j^2\right)^n$.

2. We decompose the polynomial q into irreducible factors

$$q = q_1^{\lambda_1} \cdots q_l^{\lambda_l}. \tag{38}$$

I am going to prove that q_1^n divides the inverse of the Jacobian Δ. Suppose firstly that q_1^2 does not divide $\sum_{j=1}^n p_j^2$. The inverse of the Jacobian of the transformations of (29) and (34) is equal to

$$\left\{\sum_{j=1}^n \left(\frac{p_j}{q}\right)^2\right\}^n \Delta(z_1, \cdots, z_n).$$

It follows that q_1^n divides Δ.

Suppose that q_1^2 divides $\sum_{j=1}^n p_j^2$. Without loss of generality, we may assume that q_1 does not divide p_1. The inverse of the Jacobian of the product of (30) and (34) is equal to

$$\left\{1 - 2\frac{p_1}{q} + \sum_{j=1}^n \left(\frac{p_j}{q}\right)^2\right\}^n \Delta(z_1, \cdots, z_n).$$

Evidently q_1^n divides $\Delta(z_1, \cdots, z_n)$.

Since $\sum z_j^2$ is an irreducible polynomial when $n \geqslant 3$, and Δ is of degree $2n$, we find immediately that $l = 1$ and q_1 is of degree 2. We therefore have

$$q(z_1, \cdots, z_n) = \left(a\sum_{j=1}^n z_j^2 + \cdots\right)^\lambda \tag{39}$$

and

$$\Delta(z_1, \cdots, z_n) = \rho \left(\sum_{j=1}^{n} z_j^2 \right)^n + \cdots .$$

Consequently

$$\Delta(z_1, \cdots, z_n) = \text{constant} \times \{q(z_1, \cdots, z_n)\}^{n/\lambda}. \tag{40}$$

By means of a translation (if necessary), we can assume that

$$q(z_1, \cdots, z_n) = \left(\sum_{j=1}^{n} z_j^2 + c \right)^\lambda . \tag{41}$$

3. The product of (29) and (34) is equal to

$$z_i^* = \frac{p_i}{\left(\sum p_j^2\right)/q}. \tag{42}$$

If q does not divide $\sum p_j^2$, there exists a manifold which is mapped into the point $z_i^* = 0$. This is impossible. Therefore q divides $\sum p_j^2$. By the argument that gave(41), we have

$$\frac{\sum p_j^2}{q} = \left(a \sum_{j=1}^{n} z_j^2 + \sum_{j=1}^{n} \beta_j z_j + \gamma \right)^\lambda . \tag{43}$$

Applying the same argument to the product of (30) and (34), we have immediately

$$q - 2p_k + \frac{1}{q} \sum p_j^2 = \left(\alpha_k \sum_{j=1}^{n} z_j^2 + \sum_{j=1}^{n} \beta_{kj} z_j + \gamma_k \right)^\lambda . \tag{44}$$

We suppose that $\lambda > 1$. From (41), (43), (44) we obtain

$$2z_k^* = 1 + \left(\frac{\alpha \sum z_j^2 + \sum \beta_j z_j + \gamma}{\sum z_j^2 + c} \right)^\lambda - \left(\frac{\alpha_k \sum z_j^2 + \sum \beta_{kj} z_j + \gamma_k}{\sum z_j^2 + c} \right)^\lambda . \tag{45}$$

Then we have

$$2\frac{\partial z_k^*}{\partial z_l} = \lambda \left(\frac{\alpha \sum z_j^2 + \sum \beta_j z_j + \gamma}{\sum z_j^2 + c} \right)^{\lambda-1} \frac{\partial}{\partial z_l} \left(\frac{\alpha \sum z_j^2 + \sum \beta_j z_k + \gamma}{\sum z_j^2 + c} \right)$$

$$-\lambda \left(\frac{\alpha_k \sum z_j^2 + \sum \beta_{kj} z_j + \gamma_k}{\sum z_j^2 + c} \right)^{\lambda-1} \frac{\partial}{\partial z_l} \left(\frac{\alpha_k \sum z_j^2 + \sum \beta_{kj} z_j + \gamma_k}{\sum z_j^2 + c} \right).$$

If there exists a point such that

$$\alpha\sum z_j^2+\sum\beta_j z_j+\gamma=0, \tag{46}$$

$$\alpha_k\sum z_j^2+\sum\beta_{kj} z_j+\gamma_k=0 \tag{47}$$

but

$$\sum_{j=1}^{n} z_j^2+c\neq 0,$$

then the point will make the Jacobian vanish. This violates the one-to-one relationship. Thus (46) and(47) imply

$$\sum_{j=1}^{n} z_j^2+c=0. \tag{48}$$

Consequently, (48) is a linear combination of (46) and (47). Further, $\sum(p_i/q)^2$ cannot be a constant. Thus (47) is a linear combination of (46) and (48). This is impossible when $n\geqslant 3$ because of the independence of $z_1^*,\cdots,z_n^*$.

4. We therefore have $\lambda=1$. From (41), (43), (44). with some slight modification, we may write (33) as

$$z_k^*=\frac{\sum\beta_{ij}z_j+\gamma_i}{\sum z_j^2+c}. \tag{49}$$

Since q divides $\sum\limits_{j=1}^{n} p_j^2$, we have

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\beta_{ij}z_j+\gamma_i\right)^2=\rho\left(\sum_{j=1}^{n} z_j^2+c\right). \tag{50}$$

It follows immediately that

$$\sum_{k=1}^{n}\beta_{ki}\beta_{kj}=p^2\delta_{ij}. \tag{51}$$

$$\sum_{k=1}^{n}\beta_{ki}\gamma_k=0 \tag{52}$$

and

$$\sum_{k=1}^{n}\gamma_k^2=c. \tag{53}$$

From (51) we find that

$$\frac{1}{\rho}(\beta_{ki}) = b_{ki}$$

is an orthogonal matrix. From (52) and (53) we get $\gamma_k = 0, c = 0$. Thus, by multiplying the transformations of the groups G, (33) now takes the form

$$z_i^* = \frac{\rho \sum_{j=1}^{n} b_{ij} z_j}{\sum_{j=1}^{n} z_j^2}. \tag{54}$$

It belongs evidently to the group G. Therefore, we have established the completeness of the group G when $n \geqslant 3$.

Consequently every element of G is a product of the transformations (20), (21), (22).

I should remark that when $n = 1$ the theorem is well known. When $n = 2$, it is not difficult to establish that the group is not simple, the space being a topological product of two Cauchy planes. More precisely, we use

$$x_1 x_2 - y_1 y_2 = 0$$

instead of (12). The group G may be obtained from

$$z_1^* = \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \qquad z_2^* = \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2},$$

with an additional permutation

$$z_1^* = z_2, \qquad z_2^* = z_1.$$

This is the case known as "the space of the theory of functions". Thus we have solved the problem completely.

On the Riemannian Curvature in the Space of Several Complex Variables*

§1. Introduction

Let $z = (z^1, \cdots, z^n)$ be a vector of n complex variables $z^k = x^k + iy^k$ where x^k and y^k are real numbers. Let $\mathfrak{D}$ be a bounded schlicht domain in the $2n$-dimensional space formed by x^k and $y^k (1 \leqslant k \leqslant n)$. It was established by S.BERGMANN [1] that the class $\mathfrak{L}^2 = \mathfrak{L}^2(\mathfrak{D})$ of functions $f(z)$, regular in $\mathfrak{D}$ and square integrable over $\mathfrak{D}$(i.e.,

$$\int_{\mathfrak{D}} |f(z)|^2 \dot{z} < \infty, \quad \dot{z} = \prod_{k=1}^{n} dx^k dy^k) \tag{1.1}$$

forms a HILBERT space. We can select from $\mathfrak{L}^2$ a complete orthonormal system

$$\varphi_0(z), \varphi_1(z), \cdots, \varphi_\alpha(z), \cdots. \tag{1.2}$$

By orthonormal we mean that

$$\int_{\mathfrak{D}} \varphi_\alpha(z)\overline{\varphi_\beta(z)}\dot{z} = \begin{cases} 0, & \text{if} \quad \alpha \neq \beta, \\ 1, & \text{if} \quad \alpha = \beta. \end{cases} \tag{1.3}$$

He introduces also the kernel function

$$K(z, \bar{z}) = \sum_{\alpha=0}^{\infty} \varphi_\alpha(z)\overline{\varphi_\alpha(z)}, \tag{1.4}$$

which is independent of the choice of the complete orthonormal system $\{\varphi_\alpha\}$.

We metricalize the space $\mathfrak{D}$ by introducing the Hermitian differential form, under tensor convention,

$$T_{i\bar{j}} dz^i d\bar{z}^j, \tag{1.5}$$

where

$$T_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log K(z, \bar{z}). \tag{1.6}$$

* The materials of the paper are contained in two papers (HUA[1,2]).

Sonderdruck aus: Schriftenreihe des instituts für Mathematik Bei der Deutschen Akademie der Wissenschaften zu Berlin. Heft 1 Erschienen im Akademie-Verlag. Berlin, 1957.

The Riemannian curvature of the space $\mathfrak{D}$ is defined by

$$R = \frac{R_{\bar{k}ij\bar{l}}dz^i dz^j d\bar{z}^k d\bar{z}^l}{(T_{i\bar{k}dz^i d\bar{z}^k})^2}, \tag{1.7}$$

where

$$R_{\bar{k}ij\bar{l}} = -\frac{\partial^2}{\partial z^j \partial \bar{z}^l} T_{i\bar{k}} + \left(\frac{\partial}{\partial z^j} T_{i\bar{p}}\right) \cdot T^{p\bar{q}} \left(\frac{\partial}{\partial \bar{z}^l} T_{q\bar{k}}\right) \tag{1.8}$$

and $(T^{i\bar{k}})$ is the inverse matrix of $(T_{i\bar{k}})$.

By means of variational method, Fuchs[1] proved that

$$R > 2. \tag{1.9}$$

In the first part of the paper, we shall establish a more general result by means of an elementary method. We shall assume neither the completeness, nor the orthonormality of the system φ_ν nor the boundedness of the domain $\mathfrak{D}$. In the proof, we only use the analytic property that if $f(z)$ is an analytic function, then $\frac{\partial}{\partial \bar{z}^k} f(z) = 0$. Beside this property, all the other manipulations are purely algebraic. We express also $2-R$ as a sum of squares.

In the second part, we estimate R from below. We shall prove that under certain restriction, we have

$$R \geqslant -n. \tag{1.10}$$

The restriction does not exclude the interesting case that $\mathfrak{D}$ is a transitive domain.

In the third part, as a feable generalization of celebrated Riemann theorem on conformal mapping, we prove that every non-continuable bounded domain with constant curvature can be carried onto a unit sphere by means of an analytic mapping (or pseudo-conformal mapping). Such a result seems to be the first step toward the interesting classification problem of non-continuable domain under analytic mappings.

Part I

§2. Unitary geometry

Let z be a variable over the domain $\mathfrak{D}$. For the time being we do not assume the boundedness on $\mathfrak{D}$. Let H be a non-singular Hermitian matrix with elements $H_{i\bar{j}}$ which are analytic functions of z and $\bar{z}$. The fundamental Hermitian form is given by

$$dzH\overline{dz'}, \tag{2.1}$$

where A' dcnotes the transposed matrix of A.

Let

$$z = t(w) \tag{2.2}$$

be an analytic mappıng with Jacobian

$$J = J(w) = \frac{\partial(z^1, \cdots, z^n)}{\partial(w^1, \cdots, w^n)}, \tag{2.3}$$

which is a non-singular matrix.

Then (2.2) carries (2.1) into

$$dwK\overline{dw'},$$

where

$$K = JH\bar{J}', \tag{2.4}$$

since

$$dz = dwJ. \tag{2.5}$$

Differentiating (2.4), we have

$$dK = (JdH + dJH)\overline{J'}$$

and

$$dK \cdot K^{-1} = J(dH \cdot H^{-1})J^{-1} + dJ \cdot J^{-1}.$$

Differentiating conjugately, we have

$$\bar{d}(dK \cdot K^{-1}) = J\bar{d}(dH \cdot H^{-1})J^{-1},$$

that is

$$(\bar{d}dK - dK \cdot K^{-1}\bar{d}K)K^{-1} = J(\bar{d}dH - dH \cdot H^{-1}\bar{d}H)H^{-1}J^{-1}.$$

Notice that d and $\bar{d}$ are the operations $\frac{\partial}{\partial z^j}dz^j$ and $\frac{\partial}{\partial \bar{z}^j}d\bar{z}^j$ respectively. Multiplying by (2.4), we have

$$\bar{d}dK - dK \cdot K^{-1}\bar{d}K = J(\bar{d}dH - dH \cdot H^{-1}\bar{d}H)\bar{J}'. \tag{2.6}$$

Therefore $\bar{d}dH - dH \cdot H^{-1}\bar{d}H$ is an Hermitian matrix which transforms covariantly with H. Multiplying by vector dw on the left and $\overline{dw'}$ on the right, we have

$$dw(\bar{d}dK - dK \cdot K^{-1}\bar{d}K)\overline{dw'} = dz(\bar{d}dH - dH \cdot H^{-1}\bar{d}H)\overline{dz'}. \tag{2.7}$$

In comparing with the definition (1.8), we obtain

$$R = -\frac{dz(\bar{d}dH - dHH^{-1}\bar{d}H)\overline{dz'}}{(dzH\overline{dz'})^2}. \tag{2.8}$$

Theorem 1 *Let $q(z,\bar{z})$ be an analytic function of z and $\bar{z}$ and $\overline{q(z,\bar{z})} = q(z,\bar{z})$. Then*

$$H^* = q(z,\bar{z})H$$

is again an Hermitian matrix. Let R^* be the Riemannian curvature with respect to the fundamental tensor H^*.

Then

$$qR^* = R - \frac{\bar{d}d\log q(z,\bar{z})}{dzH\overline{dz'}}.$$

Proof We have

$$\begin{aligned}
&\bar{d}dH^* - dH^* \cdot H^{*-1}\bar{d}H^* \\
=&\bar{d}(dqH + qdH) - (dqH + qdH)q^{-1}H^{-1}(\bar{d}qH + qdH) \\
=&q(\bar{d}dH - dH \cdot H^{-1}\bar{d}H) + (\bar{d}dq - q^{-1}dq\bar{d}q)H.
\end{aligned}$$

Since

$$\bar{d}d\log q = \bar{d}(q^{-1}dq) = q^{-1}(\bar{d}dq - q^{-1}dq\bar{d}q),$$

we have the theorem.

§3. Unitary geometry with fundamental tensor (1.5)

Now we take

$$H_{i\bar{j}} = T_{i\bar{j}} = \partial_i\partial_{\bar{j}}\log K = \frac{1}{K}\partial_i\partial_{\bar{j}}K - \frac{1}{K^2}\partial_i K\partial_{\bar{j}}K, \tag{3.1}$$

where $\partial_i = \dfrac{\partial}{\partial z^i}$, $\partial_{\bar{j}} = \dfrac{\partial}{\partial \bar{z}^j}$. For the present, we assume neither the completeness nor the orthonormality of the sequence $\{\varphi_\nu\}$. For assuring the meaning of $\log K$ we assume only that the functions φ_ν have no common zero in $\mathfrak{D}$.

Substituting (1.4) into (3.1) we have

$$\begin{aligned}
T_{i\bar{j}} &= \frac{1}{K^2}\sum_{\alpha,\beta=0}^{\infty}(\varphi_\alpha\bar{\varphi}_\alpha\partial_i\varphi_\beta\partial_{\bar{j}}\bar{\varphi}_\beta - \partial_i\varphi_\alpha\bar{\varphi}_\alpha\varphi_\beta\partial_{\bar{j}}\bar{\varphi}_\beta) \\
&= \frac{1}{K^2}\sum_{\alpha>\beta}(\varphi_\alpha\partial_i\varphi_\beta - \varphi_\beta\partial_i\varphi_\alpha)\overline{(\varphi_\alpha\partial_j\varphi_p - \varphi_\beta\partial_j\varphi_\alpha)},
\end{aligned} \tag{3.2}$$

and

$$dzHd\bar{z}' = \frac{1}{K^2}\sum_{\alpha>\beta}|\varphi_\alpha d\varphi_\beta - \varphi_\beta d\varphi_\alpha|^2. \tag{3.3}$$

We obtain at once

Theorem 2 *Let*

$$\varphi_0(z), \varphi_1(z), \cdots, \varphi_\alpha(z), \cdots$$

be a sequence of functions analytic in $\mathfrak{D}$ and without common zero in $\mathfrak{D}$. If the series

$$K(z,\bar{z})\sum_{\alpha=0}^{\infty}\varphi_\alpha(z)\overline{\varphi_\alpha(z)}$$

converges uniformly in any compact $\mathfrak{D}^*$ interior to $\mathfrak{D}$, the Hermitian differential form $d\,\bar{d}$ log K is positive definite (including semi-definite).

If $d\,\bar{d}\log K = 0$, from (3.3), we have

$$\varphi_\alpha d\varphi_\beta - \varphi_\beta d\varphi_\alpha = 0$$

for all α and β. That is, we have a vector $\xi = (\xi^1, \cdots, \xi^n)$such that

$$(\varphi_\alpha \partial_i \varphi_\beta - \varphi_\beta \partial_i \varphi_\alpha)\xi^i = 0, \tag{3.4}$$

that is, the rank of the $n \times \infty$ matrix

$$\begin{pmatrix} \varphi_\alpha & \partial_1\varphi_\beta - \varphi_\beta & \partial_1\varphi_\alpha \\ \vdots & \vdots & \vdots \\ \varphi_\alpha & \partial_n\varphi_\beta - \varphi_\beta & \partial_n\varphi_\alpha \end{pmatrix}_{\alpha>\beta\geqslant 0} \tag{3.5}$$

is $< n$. Notice that

$$\varphi_\alpha(\varphi_\beta\partial_i\varphi_\nu - \varphi_\nu\partial_i\varphi_\beta) + \varphi_\beta(\varphi_\nu\partial_i\varphi_\alpha - \varphi_\alpha\partial_i\varphi_\nu) + \varphi_\nu(\varphi_\alpha\partial_i\varphi_\beta - \varphi_\beta\partial_i\varphi_\alpha) = 0. \tag{3.6}$$

Then, if $\varphi_0(z)$ does not vanish at $z = z_0$, the rank of (3.5) at $z = z_0$ is equal to that of

$$\begin{pmatrix} \varphi_0 & \partial_1\varphi_\beta - \varphi_\beta & \partial_1\varphi_0 \\ \vdots & \vdots & \vdots \\ \varphi_0 & \partial_n\varphi_\beta - \varphi_\beta & \partial_n\varphi_0 \end{pmatrix}_{\beta=1,2,\cdots}$$

that is, that of

$$\begin{pmatrix} \partial_1\left(\dfrac{\varphi_\beta}{\varphi_0}\right) \\ \vdots \\ \partial_n\left(\dfrac{\varphi_\beta}{\varphi_0}\right) \end{pmatrix} \tag{3.7}$$

Theorem 3 *If the linear closure of the sequence* $\{\varphi_\nu\}$ *contains the functions* $1, z^1, \cdots, z^n$, *then the fundamental tensor* (3.3) *is always strictly definite. Consequently, if* $\{\varphi_\nu\}$ *is a complete orthonormal system, the fundamental tensor is strictly definite. By linear closure, we mean the set formed by the functions* $f(z)$ *representable by series*

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(z)$$

convergent uniformly in any $\mathfrak{D}^*$ interior to $\mathfrak{D}$.

Proof From

$$1 = \sum_{\nu=0}^{\infty} a_{0\nu}\varphi_\nu(z),$$

$$z^k = \sum_{\nu=0}^{\infty} a_{k\nu}\varphi_\nu(z), \qquad 1 \leqslant k \leqslant n,$$

we deduce that

$$\begin{aligned} dz^k &= \sum_{\nu=0}^{\infty} a_{k\nu} d\varphi_\nu(z) \\ &= \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} a_{k\nu} d\varphi_\nu \cdot a_{0\mu}\varphi_\mu - \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} a_{k\nu}\varphi_\nu a_{0\mu} d\varphi_\mu \\ &= \sum_{\nu>\mu} a_{k\nu}a_{0\mu}(\varphi_\mu d\varphi_\nu - \varphi_\nu d\varphi_\mu), \end{aligned}$$

i.e.,

$$\delta_i^k = \sum_{\nu>\mu} a_{k\nu}a_{0\mu}(\varphi_\mu \partial_i \varphi_\nu - \varphi_\nu \partial_i \varphi_\mu).$$

The rank of (3.5) is evidently equal to n. Theorem 3 is proved.

For the later usage, we prove the following corollary.

Theorem 4 *If*

$$K(z, \bar{z}) = |\xi(z)|^2 \left(1 + a\sum_{i=1}^{n} |\varphi_i(z)|^2\right)^b \qquad (a, b \text{ real}) \tag{3.8}$$

is a kernel function of a Hilbert space defined in §1, *then the Jacobian of* $\varphi_i(z)$ *does not vanish in* $\mathfrak{D}$.

Proof We have

$$d\bar{d}\ \log K(z,\bar{z}) = bd\bar{d}\log\left(1+a\sum_{i=1}^{n}|\varphi_i(z)|^2\right)$$

$$= \frac{b}{\left(1+a\sum_{i=1}^{n}|\varphi_i(z)|^2\right)^2}\left[a\sum_{i=1}^{n}|d\varphi_i|^2\left(1+a\sum_{j=1}^{n}|\varphi_j|^2\right) - a^2\sum_{i=1}^{n}d\varphi_i\bar{\varphi}_i\sum_{j=1}^{n}\varphi_j\overline{d\varphi_j}\right]$$

$$= \frac{b}{\left(1+a\sum_{i=1}^{n}|\varphi_i(z)|^2\right)^2}\left[a\sum_{i=1}^{n}|d\varphi_i|^2 + a^2\sum_{n\geqslant\nu>\mu\geqslant 1}|d\varphi_\nu\varphi_\mu - d\varphi_\mu\varphi_\nu|^2\right]$$

If the Jacobian of $\varphi_i(z)$, i.e.,

$$|\partial_j\varphi_i(z)|_{1\leqslant i,j\leqslant n}$$

vanishes, at a point $z=z_0$, then we hace a non-zero vector $\xi=(\xi^1,\cdots.\xi^n)$ such that

$$\partial_j\varphi_i(z)\xi^j=0,$$

at $z=z_0$.

That is, the Hermitian form

$$d\bar{d}\log K(\bar{z}.z)$$

vanishes at $z=z_0$ for $dz^i=\xi^i$. Combining with Theorem 3, we have Theorem 4.

§4. Riemannian curvature $\leqslant 2$

We arrange (α,β) $(\alpha>\beta)$ according to the following order: If $\alpha<\alpha'$, we put (α,β) preceding (α',β') and the relation is denoted by $(\alpha,\beta)<(\alpha',\beta')$: and if $\alpha=\alpha'$ but $\beta<\beta'$, then we put (α,β) preceding $(\alpha.\beta')$ and also denoted by $(\alpha.\beta)<(\alpha',\beta')$. After the arrangement we numerate the indices by $\gamma=0,1,2,3,\cdots$; more precisely $\gamma=\frac{1}{2}\alpha(\alpha-1)+\beta$. Our fundamental tensor can be written by

$$T_{i\bar{j}}dz^i d\bar{z}^j = \frac{1}{K^2}\sum_{\gamma=0}^{\infty}u_\gamma\bar{u}_\gamma \tag{4.1}$$

and

$$T_{i\bar{j}} = \frac{1}{K^2}\sum_{\gamma=0}^{\infty}u^i_\gamma\bar{u}^j_\gamma.$$

where

$$u_\gamma = \varphi_\alpha d\varphi_\beta - \varphi_\beta d\varphi_\alpha \tag{4.2}$$

and

$$u_\gamma^i = \varphi_\alpha \partial_i \varphi_\beta - \varphi_\beta \partial_i \varphi_\alpha. \tag{4.3}$$

Taking $q = K^2$ in Theorem 1, and

$$T_{i\bar{j}}^* = \sum_{\gamma=0}^{\infty} u_\gamma^i \bar{u}_\gamma^j, \tag{4.4}$$

we have then

$$qR^* = R - \frac{\bar{d}d\log K^2}{\bar{d}d\log K} = R - 2. \tag{4.5}$$

The aim of the present section is to prove that $R^* \leqslant 0$, that is

Theorem 5 *Let $\{\varphi_\nu\}$, $\nu = 0, 1, 2, \cdots$, be a sequence of analytic functions in a domain $\mathfrak{D}$, and $\{\varphi_\nu\}$ have no common zero in $\mathfrak{D}$. Suppose that*

$$K(z, \bar{z}) = \sum_{\nu=0}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(z)}$$

converges uniformly in any compact domain interior to $\mathfrak{D}$, and that $\{\varphi_\nu/\varphi_0\}$, $\nu = 1, 2, \cdots$, contains n independent functions. Then the Riemannian curvature of the space $\mathfrak{D}$ defined by the metric

$$\bar{d}d\log K(z, \bar{z}) \tag{4.6}$$

does not exceed 2.

Proof Let $H^* = (T_{i\bar{j}}^*)$ and

$$N = \begin{pmatrix} u_\gamma^1 \\ \vdots \\ u_\gamma^n \end{pmatrix}_{\gamma=0,1,\cdots}$$

Then, evidently we have

$$H^* = N\bar{N}',$$

and

$$\bar{d}dH^* - dH^* \cdot H^{*-1}\bar{d}H^* = dN(I - \bar{N}'(N\bar{N}')^{-1}N)\overline{dN'}.$$

Substituting into (2.8), we have

$$R^* = -\frac{dzdN(I - \bar{N}'(N\bar{N}')^{-1}N)\overline{dN'}d\bar{z}'}{((dzN)(\overline{dzN})')^2}. \tag{4.7}$$

Since

$$I - \bar{N}'(N\bar{N}')^{-1}N = (I - \bar{N}'(N\bar{N}')^{-1}N)^2$$
$$= (I - \bar{N}'(N\bar{N}')^{-1}N)\overline{(I - \bar{N}'(N\bar{N}')^{-1}N)'}$$

is a positive definite Hermitian matrix, we have therefore $R^* \leqslant 0$.

Remark Theorem 5 is "best possible". In fact for n=2, and taking six functions

$$A, z^1, z^2, a_i(z^1)^2 + 2b_i z^1 z^2 + c_i(z^2)^2, \qquad i = 1, 2, 3,$$

as our $\{\varphi_\nu\}$, we can easily calculate that, at $z = 0$,

$$2 - R = 4A^2 \frac{\sum_{i=1}^{3} |a_i(dz^1)^2 + 2b_i dz^1 dz^2 + c_i(dz^2)^2|^2}{(|dz^1|^2 + |dz^2|^2)^2}. \tag{4.8}$$

We determined (Hua[1]) also the extremal case, i.e. $R = 2$, and proved (Hua[1]) that if the linear closure of the sequence $\{\varphi_\nu\}$ contains

$$1, z^1, \cdots, z^n, z^i z^j \quad (1 \leqslant i \leqslant j \leqslant n),$$

then we hace

$$R < 2. \tag{4.9}$$

§5. Expression of 2−*R* as a sum of squares

Theorem 6 *We use* T^* *to denote the determinant of* $T^*_{i\bar{j}}$, *and*

$$x_\gamma = \partial_j u^i_\gamma dz^i dz^j. \tag{5.1}$$

Then

$$-T^* R^*_{\bar{k}ij\bar{l}} dz^i dz^j d\bar{z}^k d\bar{z}^l = \sum_{\gamma_1 > \gamma_2 > \cdots > \gamma_{n+1}} |P_{\gamma_1, \cdots, \gamma_{n+1}}|^2, \tag{5.2}$$

where

$$P_{\gamma_1, \cdots, \gamma_{n+1}} = \begin{vmatrix} u^1_{\gamma_1} & \cdots & u^n_{\gamma_1} & x_{\gamma_1} \\ \vdots & & \vdots & \vdots \\ u^1_{\gamma_{n+1}} & \cdots & u^n_{\gamma_{n+1}} & x_{\gamma_{n+1}} \end{vmatrix}. \tag{5.3}$$

Proof　1. First let us consider

$$
\begin{aligned}
A =& T^*\partial_j\partial_{\bar l}T^*_{i\bar k}dz^i dz^j d\bar z^k d\bar z^l\\
=& \sum_{\alpha_1=0}^{\infty}\cdots\sum_{\alpha_n=0}^{\infty}\begin{vmatrix} u^1_{\alpha_1} & \overline{u^1_{\alpha_1}} & \cdots & u^1_{\alpha_1} & \overline{u^n_{\alpha_1}}\\ \vdots & \vdots & & \vdots & \vdots\\ u^n_{\alpha_n} & \overline{u^1_{\alpha_n}} & \cdots & u^n_{\alpha_n} & \overline{u^n_{\alpha_n}}\end{vmatrix}\sum_{\beta=0}^{\infty}x_\beta\bar x_\beta\\
=& \sum_{\alpha_1=0}^{\infty}\cdots\sum_{\alpha_n=0}^{\infty}u^1_{\alpha_1}\cdots u^n_{\alpha_n}\begin{vmatrix}\overline{u^1_{\alpha_1}} & \cdots & \overline{u^n_{\alpha_n}}\\ \vdots & & \vdots\\ \overline{u^1_{\alpha_n}} & \cdots & \overline{u^n_{\alpha_n}}\end{vmatrix}\sum_{\beta=0}^{\infty}x_\beta\bar x_\beta\\
=& \sum_{\alpha_1>\alpha_2>\cdots>\alpha_n}\begin{vmatrix} u^1_{\alpha_1} & \cdots & u^n_{\alpha_n}\\ \vdots & & \vdots\\ u^1_{\alpha_n} & \cdots & u^n_{\alpha_n}\end{vmatrix}\overline{\begin{vmatrix} u^1_{\alpha_1} & \cdots & u^n_{\alpha_n}\\ \vdots & & \vdots\\ u^1_{\alpha_n} & \cdots & u^n_{\alpha_n}\end{vmatrix}}\sum_{\beta=0}^{\infty}x_\beta\bar x_\beta,
\end{aligned}
\tag{5.4}
$$

by (4.4).

Write

$$
P_{\gamma_2,\cdots,\gamma_{n+1}} = \begin{vmatrix} u^1_{\gamma_1} & \cdots & u^n_{\gamma_1} & x_{\gamma_1}\\ \vdots & & \vdots & \vdots\\ u^1_{\gamma_{n+1}} & \cdots & u^n_{\gamma_{n+1}} & x_{\gamma_{n+1}}\end{vmatrix} = \sum_{i=1}^{n+1}(-1)^{n+i}P_i x_{\gamma_i}. \tag{5.5}
$$

If

$$
\alpha_1>\alpha_2>\cdots>\alpha_r>\beta>\alpha_{r+1}>\cdots>\alpha_n,
$$

we take $\alpha_1=\gamma_1,\cdots,\alpha_r=\gamma_r,\beta=\gamma_{r+1},\alpha_{r+1}=\gamma_{r+2},\cdots,\alpha_n=\gamma_{n+1}$, then

$$
A=\sum_{\gamma_1>\gamma_2>\cdots>\gamma_{n+1}}\sum_{i=1}^{n+1}p_i\bar p_i x_{\gamma_i}\bar x_{\gamma_i}+B, \tag{5.6}
$$

where

$$
B=\sum_{r=1}^{n}\sum_{\alpha_1>\cdots>\alpha_n}\begin{vmatrix} u^1_{\alpha_1} & \cdots & u^n_{\alpha_1}\\ \vdots & & \vdots\\ u^1_{\alpha_n} & \cdots & u^n_{\alpha_n}\end{vmatrix}\overline{\begin{vmatrix} u^1_{\alpha_1} & \cdots & u^n_{\alpha_1}\\ \vdots & & \vdots\\ u^1_{\alpha_n} & \cdots & u^n_{\alpha_n}\end{vmatrix}}x_{\alpha_r}\bar x_{\alpha_r}. \tag{5.7}
$$

2. Next we consider

$$
\begin{aligned}
C =& -T^*\partial_j T^*_{i\bar p}T^{*p\bar q}\partial_{\bar l}T^*_{q\bar k}dz^i dz^j d\bar z^k d\bar z^l\\
=& -\sum_{\beta_1=0}^{\infty}\sum_{\beta_2=0}^{\infty}\bar u^p_{\beta_1}(T^*T^{*p\bar q})u^q_{\beta_2}x_{\beta_1}\bar x_{\beta_2}.
\end{aligned}
\tag{5.8}
$$

As in 1, we obtain

$$
\begin{aligned}
&T^{*}T^{*p\bar{q}}\\
&=(-1)^{p+q}\sum_{\alpha_1=0}^{\infty}\cdots\\
&\sum_{\alpha_{n-1}=0}^{\infty}\begin{vmatrix}
u^1_{\alpha_1} & \overline{u^1_{\alpha_1}} & \cdots & u^1_{\alpha_1} & \overline{u^{p-1}_{\alpha_1}} & u^1_{\alpha_1} & \overline{u^{p+1}_{\alpha_1}} & \cdots & u^1_{\alpha_1} & \overline{u^n_{\alpha_1}}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
u^{q-1}_{\alpha_{q-1}} & \overline{u^1_{\alpha_{q-1}}} & \cdots & u^{q-1}_{\alpha_{q-1}} & \overline{u^{p-1}_{\alpha_{q-1}}} & u^{q-1}_{\alpha_{q-1}} & \overline{u^{p+1}_{\alpha_{q-1}}} & \cdots & u^{q-1}_{\alpha_{q-1}} & \overline{u^n_{\alpha_{q-1}}}\\
u^{q+1}_{\alpha_q} & \overline{u^1_{\alpha_q}} & \cdots & u^{q+1}_{\alpha_q} & \overline{u^{p-1}_{\alpha_q}} & u^{q+1}_{\alpha_q} & \overline{u^{p+1}_{\alpha_q}} & \cdots & u^{q+1}_{\alpha_q} & \overline{u^n_{\alpha_q}}\\
u^n_{\alpha_{n-1}} & \overline{u^1_{\alpha_{n-1}}} & \cdots & u^n_{\alpha_{n-1}} & \overline{u^{p-1}_{\alpha_{n-1}}} & u^n_{\alpha_{n-1}} & \overline{u^{p+1}_{\alpha_{n-1}}} & \cdots & u^n_{\alpha_{n-1}} & \overline{u^n_{\alpha_{n-1}}}
\end{vmatrix}\\
&=(-1)^{p+q}\sum_{\alpha_1=0}^{\infty}\cdots\sum_{\alpha_{n-1}=0}^{\infty}u^1_{\alpha_1}\cdots u^{q-1}_{\alpha_{q-1}}u^{q+1}_{\alpha_q}\cdots u^n_{\alpha_{n-1}}\\
&\quad\times\begin{vmatrix}
\overline{u^1_{\alpha_1}} & \cdots & \overline{u^{p-1}_{\alpha_1}} & \overline{u^{p+1}_{\alpha_1}} & \cdots & \overline{u^n_{\alpha_1}}\\
\vdots & & \vdots & \vdots & & \vdots\\
\overline{u^1_{\alpha_{n-1}}} & \cdots & \overline{u^{p-1}_{\alpha_{n-1}}} & \overline{u^{p+1}_{\alpha_{n-1}}} & \cdots & \overline{u^n_{\alpha_{n-1}}}
\end{vmatrix}\\
&=(-1)^{p+q}\sum_{\alpha_1>\alpha_2>\cdots>\alpha_{n-1}}\begin{vmatrix}
u^1_{\alpha_1} & \cdots & u^{q-1}_{\alpha_1} & u^{q+1}_{\alpha_1} & \cdots & u^n_{\alpha_1}\\
\vdots & & \vdots & \vdots & & \vdots\\
u^1_{\alpha_{n-1}} & \cdots & u^{q-1}_{\alpha_{n-1}} & u^{q+1}_{\alpha_{n-1}} & \cdots & u^n_{\alpha_{n-1}}
\end{vmatrix}\\
&\quad\times\overline{\begin{vmatrix}
u^1_{\alpha_1} & \cdots & u^{p-1}_{\alpha_1} & u^{p+1}_{\alpha_1} & \cdots & u^n_{\alpha_1}\\
\vdots & & \vdots & \vdots & & \vdots\\
u^1_{\alpha_{n-1}} & \cdots & u^{p-1}_{\alpha_{n-1}} & u^{p+1}_{\alpha_{n-1}} & \cdots & u^n_{\alpha_{n-1}}
\end{vmatrix}}.
\end{aligned}
\tag{5.9}
$$

Therefore, we have

$$\begin{aligned}
&\sum_{\beta_1,\beta_2=0}^{\infty} T^*T^{*p\bar{q}}\bar{u}^p_{\beta_1}u^q_{\beta_2}x_{\beta_1}\bar{x}_{\beta_2}\\
&=\sum_{\beta_1,\beta_2=0}^{\infty}\sum_{\alpha_1>\alpha_2>\cdots>\alpha_{n-1}}\begin{vmatrix} u^1_{\beta_2} & \cdots & u^n_{\beta_2}\\ u^1_{\alpha_1} & \cdots & u^n_{\alpha_1}\\ \vdots & & \vdots\\ u^1_{\alpha_{n-1}} & \cdots & u^n_{\alpha_{n-1}}\end{vmatrix}\overline{\begin{vmatrix} u^1_{\beta_1} & \cdots & u^n_{\beta_1}\\ u^1_{\alpha_1} & \cdots & u^n_{\alpha_1}\\ \vdots & & \vdots\\ u^1_{\alpha_{n-1}} & \cdots & u^n_{\alpha_{n-1}}\end{vmatrix}}x_{\beta_1}\bar{x}_{\beta_2}\\
&=B+\sum_{\alpha_1>\cdots>\alpha_{n-1}}\sum_{\beta_1\neq\beta_2}\begin{vmatrix} u^1_{\beta_2} & \cdots & u^n_{\beta_2}\\ u^1_{\alpha_1} & \cdots & u^n_{\alpha_1}\\ \vdots & & \vdots\\ u^1_{\alpha_{n-1}} & \cdots & u^n_{\alpha_{n-1}}\end{vmatrix}\overline{\begin{vmatrix} u^1_{\beta_1} & \cdots & u^n_{\beta_1}\\ u^1_{\alpha_1} & \cdots & u^n_{\alpha_1}\\ \vdots & & \vdots\\ u^1_{\alpha_{n-1}} & \cdots & u^n_{\alpha_{n-1}}\end{vmatrix}}x_{\beta_1}\bar{x}_{\beta_2}.
\end{aligned}\tag{5.10}$$

If $\alpha_i=\beta_1$ or β_2, the corresponding term vanishes. Therefore we may assume that α_i equals neither β_1 nor β_2. Suppose $\beta_1>\beta_2$, we arrange

$$\alpha_1>\cdots>\alpha_u>\beta_1>\alpha_{u+1}>\cdots>\alpha_{u+v}>\beta_2>\alpha_{u+v+1}>\cdots>\alpha_{n-1}.$$

By taking

$$\gamma_i=\alpha_i(1\leqslant i\leqslant u),\quad \gamma_{u+1}=\beta_1,\quad \gamma_{u+1+j}=\alpha_{u+j}\quad(1\leqslant j\leqslant u),$$

$$\gamma_{u+v+2}=\beta_2,\quad \gamma_{u+v+2+k}=\alpha_{u+v+k}\quad(1\leqslant k\leqslant n-u-v-1),$$

the corresponding term can be written as

$$(-1)^{u+u+v}p_{u+1}\bar{p}_{u+v+2}x_{\gamma u+1}\bar{x}_{\gamma u+\tau+2}.$$

If $\beta_1<\beta_2$, we have a similar result.

Therefore

$$
\begin{aligned}
A+C &= \sum_{\gamma_1>\gamma_2>\cdots>\gamma_{n+1}} \left(\sum_{i=1}^{n} p_i \bar{p}_i x_{\gamma_i} \bar{x}_{\gamma_i} + \sum_{j\neq k} (-1)^{j+k} p_j \bar{p}_k x_{\gamma_i} \bar{x}_{\gamma_k} \right) \\
&= \sum_{\gamma_1>\gamma_2>\cdots>\gamma_{n+1}} \left(\sum_{j=1}^{n} (-1)^{j-1} p_i x_{\gamma_j} \right) \left(\sum_{k=1}^{n} (-1)^{k-1} \bar{p}_k \bar{x}_{\gamma_k} \right) \\
&= \sum_{\gamma_1>\gamma_2>\cdots>\gamma_{n+1}} \begin{vmatrix} u_{\gamma_1}^1 & \cdots & u_{\gamma_1}^n & x_{\gamma_1} \\ \vdots & & \vdots & \vdots \\ u_{\gamma_{n+1}}^1 & \cdots & u_{\gamma_{n+1}}^n & x_{\gamma_{n+1}} \end{vmatrix} \overline{\begin{vmatrix} u_{\gamma_1}^1 & \cdots & u_{\gamma_1}^n & x_{\gamma_1} \\ \vdots & & \vdots & \vdots \\ u_{\gamma_{n+1}}^1 & \cdots & u_{\gamma_{n+1}}^n & x_{\gamma_{n+1}} \end{vmatrix}},
\end{aligned}
\tag{5.11}
$$

the theorem is proved.

§6. Ricci tensor

Let the determinant of the Hermitian form $\bar{d}d \log K$ be denoted by

$$D = |\partial_i \partial_{\bar{j}} \log K|. \tag{6.1}$$

From (4.1), we have

$$
\begin{aligned}
D &= \frac{1}{K^{2n}} \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} \begin{vmatrix} u_{\alpha_1}^1 & \overline{u_{\alpha_1}^1} & \cdots & u_{\alpha_1}^1 & \overline{u_{\alpha_2}^n} \\ \vdots & \vdots & & \vdots & \vdots \\ u_{\alpha_n}^n & \overline{u_{\alpha_n}^1} & \cdots & u_{\alpha_n}^n & \overline{u_{\alpha_n}^n} \end{vmatrix} \\
&= \frac{1}{K^{2n}} \sum_{\alpha_1>\alpha_2>\cdots>\alpha_n} \Phi_{\alpha_1,\cdots,\alpha_n}(z) \overline{\Phi_{\alpha_1,\cdots,\alpha_n}(z)},
\end{aligned}
\tag{6.2}
$$

where

$$\Phi_{\alpha_1,\cdots,\alpha_n}(z) = \begin{vmatrix} u_{\alpha_1}^1 & \cdots & u_{\alpha_1}^n \\ \vdots & & \vdots \\ u_{\alpha_n}^1 & \cdots & u_{\alpha_n}^n \end{vmatrix}. \tag{6.3}$$

We arrange the order of $(\alpha_1, \cdots, \alpha_n)$ $(\alpha_1 > \alpha_2 > \cdots > \alpha_n)$ in the following way: if $\alpha_1 = \beta_1, \cdots, \alpha_{r-1} = \beta_{r-1}$ but $\alpha_r > \beta_r$, we define that $(\beta_1, \cdots, \beta_n)$ precedes $(\alpha_1, \cdots, \alpha_n)$. We numerate this sequence by numbers 0, 1, 2, $\cdots$, ω, $\cdots$ Then we

have

$$D=\frac{1}{K^{2n}}\sum_{\omega=0}^{\infty}\Phi_{\omega}(z)\overline{\Phi_{\omega}(z)}. \tag{6.4}$$

Similar to the method given in §3, the Hermitian form defined by the Ricci tensor can be written as

$$\begin{aligned}-\partial_i\partial_{\bar{j}}\log D dz^i d\bar{z}^j = &-\sum_{\omega>\eta}|\Phi_\omega d\Phi_\eta-\Phi_\eta d\Phi_\omega|^2\Big/\left(\sum_{\omega=0}^{\infty}|\Phi_\omega|^2\right)^2\\ &+2n\sum_{\alpha>\beta}|\varphi_\alpha d\varphi_\beta-\varphi_\beta d\varphi_\alpha|^2\Big/\left(\sum_{\alpha=0}^{\infty}|\varphi_\alpha|^2\right)^2.\end{aligned} \tag{6.5}$$

Part II

§7. Space with further restriction

In part II we add a further restriction that

$$\sum_{\nu=0}^{\infty}\Phi_\nu(z)\overline{\Phi_\nu(z)}=c\left(\sum_{\nu=0}^{\infty}\varphi_\nu(z)\overline{\varphi_\nu(z)}\right)^{2n+1} \tag{7.1}$$

where c is a constant and Φ_ν is defined in §6 or Φ_ν runs over all the n-rowed minors of the infinite matrix

$$\begin{pmatrix}\varphi_\alpha\partial_1\varphi_\beta-\varphi_\beta\partial_1\varphi_\alpha\\ \vdots\\ \varphi_\alpha\partial_n\varphi_\beta-\varphi_\beta\partial_n\varphi_\alpha\end{pmatrix}_{\alpha>\beta,\alpha,\beta=0,1,2,\cdots} \tag{7.2}$$

First of all, let us give some explanation about the condition (7.1). It is known [cf. (6.1)] that

$$D=|\partial_i\partial_{\bar{j}}\log K|=\frac{1}{K^{2n}}\sum_{\gamma=0}^{\infty}|\Phi_\gamma|^2$$

is the volume density of the Riemannian space. (7.1) can be written as

$$cK=D, \tag{7.3}$$

that is, the kernel function is proportional to the volume density. Consequently the Riemannian space so obtained is an Einstein space.

If the space $\mathfrak{D}$ is transitive, let

$$z=t(w)$$

be a transformation carrying $\mathfrak{D}$ onto itself. Then the kernel function $K(z, \bar{z})$ of the space satisfies

$$K(z, \bar{z}) = K(w, \bar{w})|J(w)|^2$$

and also

$$D(z, \bar{z}) = D(w, \bar{w})|J(w)|^2.$$

Since

$$\frac{K(z, \bar{z})}{D(z, \bar{z})} = \frac{K(w, \bar{w})}{D(w, \bar{w})}$$

for all w,

$$\frac{K(z, \bar{z})}{D(z, \bar{z})}$$

is a constant. Therefore, for the space $\mathfrak{D}$ admitting a transitive group of analytic transformations, the geometry defined in §1 satisfies our condition (7.1).

Now let us go back to our original situation. We introduce several lemmas which are useful for simplification:

Theorem 7 *Let $z = t(w)$ be an analytic mapping with Jacobian $J(w)$. Let*

$$\psi_\nu(w) = \varphi_\nu(t(w))J(w). \tag{7.4}$$

We construct $\Psi_\nu(w)$ from $\psi_\nu(w)$ as $\Phi_\nu(z)$ from $\varphi_\nu(z)$. Then, the relation (7.1) holds, if we replace $\varphi_\nu(z)$ by $\psi_\nu(w)$ and $\Phi_\nu(z)$ by $\Psi_\nu(w)$.

Proof Since

$$\psi_\alpha(w)\frac{\partial}{\partial w^i}\psi_\beta(w) - \psi_\beta(w)\frac{\partial}{\partial w^i}\psi_\alpha(w) = \frac{\partial z^j}{\partial w^i}(\varphi_\alpha(z)\partial_j\varphi_\beta(z) - \varphi_\beta(z)\partial_j\varphi_\alpha(z))(J(w))^2$$

and

$$\Psi_\nu(w) = \det\left(\frac{\partial z^j}{\partial w^i}\right)\Phi_\nu(z)(J(w))^{2n} = \Phi_\nu(z)(J(w))^{2n+1}.$$

we have

$$\begin{aligned}\sum_{\nu=0}^{\infty}\Psi_\nu(w)\overline{\Psi_\nu(w)} &= |J(w)|^{2(2n+1)}\sum_{\nu=0}^{\infty}\Phi_\nu(z)\overline{\Phi_\nu(z)} \\ &= c|J(w)|^{2(2n+1)}\left(\sum_{\nu=0}^{\infty}\varphi_\nu(z)\overline{\varphi_\nu(z)}\right)^{2n+1} \\ &= c\left(\sum_{\nu=0}^{\infty}\psi_\nu(w)\overline{\psi_\nu(w)}\right)^{2n+1}\end{aligned}$$

Theorem 8 *If*

$$\psi_\nu(z) = \sum_{\mu=0}^{\infty}a_{\nu\mu}\varphi_\mu(z), \quad \nu = 0, 1, 2, \cdots,$$

where $(a_{\nu\mu}) = U$ *is an unitary matrix, the conclusion of Theorem 7 is again true.*

Proof From

$$\begin{aligned}
&\psi_\alpha(z)\partial_i\psi_\beta(z) - \psi_\beta(z)\partial_i\psi_\alpha(z)\\
=&\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} a_{\alpha\nu}a_{\beta\mu}(\varphi_\nu\partial_i\varphi_\mu - \varphi_\mu\partial_i\varphi_\nu)\\
=&\sum_{\nu>\mu}(a_{\alpha\nu}a_{\beta\mu} - a_{\alpha\mu}a_{\beta_\nu})(\varphi_\nu\partial_i\varphi_\mu - \varphi_\mu\partial_i\varphi_\nu),
\end{aligned}$$

we deduce that

$$\begin{pmatrix} \psi_\alpha\partial_1\psi_\beta - \psi_\beta\partial_1\psi_\alpha \\ \vdots \\ \psi_\alpha\partial_n\psi_\beta - \psi_\beta\partial_n\psi_\alpha \end{pmatrix} = \begin{pmatrix} \varphi_\nu\partial_1\varphi_\mu - \varphi_\mu\partial_1\varphi_\nu \\ \vdots \\ \varphi_\nu\partial_n\varphi_\mu - \varphi_\mu\partial_n\varphi_\nu \end{pmatrix}(A_{\nu\mu,\alpha\beta}), \tag{7.5}$$

where

$$A_{\mu\nu,\alpha\beta} = a_{\alpha\nu}a_{\beta\mu} - a_{\alpha\mu}a_{\beta\nu}.$$

Since

$$\begin{aligned}
\sum_{\alpha>\beta} A_{\nu\mu,\alpha\beta}\bar{A}_{\nu_1\mu_1,\alpha\beta} =& \frac{1}{2}\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}(a_{\alpha_\nu}a_{\beta\mu} - a_{\alpha\mu}a_{\beta\nu})(\bar{a}_{\alpha\nu_1}\bar{a}_{\beta\mu_1} - \bar{a}_{\alpha\mu_1}\bar{a}_{\beta\nu_1})\\
=& \frac{1}{2}(\delta_{\nu\nu_1}\delta_{\mu\mu_1} - \delta_{\mu\nu_1}\delta_{\nu\mu_1} - \delta_{\nu\mu_1}\delta_{\mu\nu_1} + \delta_{\mu\mu_1}\delta_{\nu\nu_1})\\
=& \begin{cases} 1, & \text{if} \quad \nu = \nu_1, \quad \mu = \mu_1, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

the matrix $(A_{\nu\mu,\alpha\beta})$ is unitary.

Since the n-rowed minors of (7.5) have also unitary relation, we have

$$\sum_{\nu=0}^{\infty} \Psi_\nu(z)\overline{\Psi_\nu(z)} = \sum_{\nu=0}^{\infty} \Phi_\nu(z)\overline{\Phi_\nu(z)},$$

which establishes Theorem 8.

§8. Reductions

Now we are going to perform some simplifications which do not restrict the generality.

We may suppose that $\mathfrak{D}$ contains the origin, $\varphi_0(0) \neq 0$, and that the Jacobian of $\varphi_i/\varphi_0 (1 \leqslant i \leqslant n)$ does not vainsh at the origin. Consider the expansion

$$\varphi_\nu(z) = a_0^\nu + \sum_{i=1}^{n} a_i^\nu z^i + \sum_{i,j=1}^{n} a_{ij}^\nu z^i z^j + \cdots , \tag{8.1}$$

where $a_{ij}^{\nu} = a_{ji}^{\nu}$. Since $\sum_{\nu=0}^{\infty}|\varphi_\nu(0)|^2$ converges, we have a unitary matrix $U = (u_{\nu\mu})_{\nu,\mu=0,1,2,\cdots}$ such that

$$(a_0^0, a_0^1, \cdots, a_0^\nu, \cdots)U = (\varrho, 0, \cdots, 0, \cdots),$$

where

$$\varrho^2 = \sum_{\nu=0}^{\infty}|\varphi_\nu(0)|^2.$$

Consider

$$\sum_{\nu=0}^{\infty}\varphi_\mu(z)u_{\nu\mu}$$

instead of $\varphi_\mu(z)$, we may assume, without loss of generality, that

$$\varphi_0(0) = a_0^0 \neq 0, \quad \varphi_\nu(0) = 0 \quad (\nu > 0).$$

Consider the transformation

$$\left.\begin{aligned} w^1 &= (a_0^0)^{-1}\int_0^{z^1}\varphi_0(t, z^2, \cdots, z^n)dt, \\ w^k &= z^k \quad \text{for} \quad 2 \leqslant k \leqslant n. \end{aligned}\right\} \tag{8.2}$$

The transformation carries the origin into the origin and its Jacobian

$$\frac{\partial w}{\partial z} = (a_0^0)^{-1}\varphi_0(z)$$

equals 1 at the origin. By theorem 7, with $\psi_0(w) = \varphi_0(z)J(w) = a_0^0$, we may assume without loss of generality that

$$\varphi_0(z) = A, \quad \varphi_\nu(0) = 0$$

for $\nu > 0$. Notice that the property "the Jacobian of $\varphi_k/\varphi_0 (1 \leqslant k \leqslant n)$ does not vanish at the origin" still holds.

We write

$$\varphi_\nu(z) = l_\nu(z) + q_\nu(z), \quad \nu > 0,$$

where $l_\nu(z)$ are linear terms of $\varphi_\nu(z)$ and $q_\nu(z)$ contain neither linear nor constant terms. Since the Jacobian of $\varphi_k/\varphi_0 (1 \leqslant k \leqslant n)$ does not vanish at the origin, therefore

$$l_k(z) \qquad (1 \leqslant k \leqslant n)$$

are linearly independent forms. There is a unitary matrix

$$V = (v_{\nu\mu})\nu, \quad \mu = 1, 2, \cdots$$

such that

$$\sum_{\mu=1}^{\infty} \varphi_\nu(z) v_{\nu\mu}, \quad \mu = n+1, n+2, \cdots$$

have no linear terms. Therefore, from now on, we may assume that, for $\nu > n$,

$$\varphi_\nu(z) = q_\nu(z),$$

which are power series without constant and linear terms.

Hereafter, we use $q(z)$ to denote a power series without constant and linear terms and they are not necessarily the same at each occurrence.

Let $w^i = l_i(z) = \sum_{j=1}^{n} a_j^i z^j$. By Theorem 7 we obtain

$$\psi_0(w) = Aa, \quad \psi_i(w) = (w^i + q_i(w))a, \quad 1 \leqslant i \leqslant n,$$

$$\psi_\nu(w) = q_\nu(w)a, \quad \nu > n,$$

where $a = \det(a_j^i)^{-1}$. If we replace $\varphi_\alpha(z)$ by $\varphi_\alpha(z)a(\alpha = 0, 1, 2, \cdots)$, the relation (7.1) becomes

$$\sum_{\nu=0}^{\infty} \Phi_\nu(z)\overline{\Phi_\nu(z)} = c_1 \left(\sum_{\alpha=0}^{\infty} \varphi_\alpha(z)\overline{\varphi_\alpha(z)} \right)^{2n+1}$$

where c_1 is again a constant. Therefore without loss of generality, we may assume that

$$\varphi_0(z) = A, \quad \varphi_i(z) = z^i + q_i(z), \quad 1 \leqslant i \leqslant n,$$
$$\varphi_\nu(z) = q_\nu(z), \quad \nu > n. \tag{8.3}$$

Next let us study the relation (7.1). The right hand side does not contain those terms which are functions of $\bar{z}$ and independent of z. Such terms may occur on the left if $\Phi_\nu(z)$ contain constant terms. That is, it can only happen in $|\Phi_0(z)|^2$, where

$$\Phi_0 = \begin{vmatrix} \varphi_0\partial_1\varphi_1 - \varphi_1\partial_1\varphi_0 & \cdots & \varphi_0\partial_1\varphi_n - \varphi_n\partial_1\varphi_0 \\ \vdots & & \vdots \\ \varphi_0\partial_n\varphi_1 - \varphi_1\partial_n\varphi_0 & \cdots & \varphi_0\partial_n\varphi_n - \varphi_n\partial_n\varphi_0 \end{vmatrix}$$
$$= A^n \begin{vmatrix} \partial_1\varphi_1 & \cdots & \partial_1\varphi_n \\ \vdots & & \vdots \\ \partial_n\varphi_1 & \cdots & \partial_n\varphi_n \end{vmatrix}.$$

Since the right hand side of (7.1) contains no term, different from constant and independent of z, we see that Φ_0 must be a constant. Consequently

$$\begin{vmatrix} \partial_1\varphi_1 & \cdots & \partial_1\varphi_n \\ \vdots & & \vdots \\ \partial_n\varphi_1 & \cdots & \partial_n\varphi_n \end{vmatrix} = 1.$$

Taking $\varphi_i(z) = w^i$ in theorem 7, we may assume without loss of generality that

$$\left.\begin{array}{l} \varphi_0(z) = A, \quad \varphi_i(z) = z_i, \quad i = 1, 2, \cdots, n; \\ \varphi_\nu(z) = q_\nu(z), \quad \nu = n+1, \quad n+2, \cdots \end{array}\right\} \tag{8.4}$$

Applying Theorem 7 again, with $z = A\ w$, the functions so obtained are all divided by A^{n+1}, then without loss of generality we may assume that $A = 1$. Comparing coefficients we may assume also that $c_1 = 1$.

§9. Lower bound of the curvature

Theorem 9 *Under the same assumption as in Theorem 5, we assume further that* $\{\varphi_\nu\}$ *satisfied* (7.1), *we have*

$$R \geqslant -n. \tag{9.1}$$

Proof We may assume that $\mathfrak{D}$ contains the origin and without loss of generality we shall prove (9.1) only at $z = 0$. Since the manifold defined by the Jacobian of

$$\varphi_j/\varphi_0, \quad 1 \leqslant i \leqslant n$$

vanishing is of lower dimension, it is enough to assume that the Jacobian does not vanish at the origin. By the argument used in §8, we may assume that

$$\varphi_0(z) = 1, \quad \varphi_k(z) = z^k \quad (1 \leqslant k \leqslant n)$$
$$\varphi_\nu(z) = q_\nu(z), \quad \nu = n+1, n+2, \cdots .$$

We substitute into (7.1) and consider the terms in (7.1) linear both in z and $\bar{z}$, the right hand side gives

$$(2n+1)\sum_{i=1}^{n} |z^i|^2,$$

and the left gives

$$\sum_{i>j}(|z^i|^2 + |z^j|^2) + \sum_{i=1}^{n}\sum_{\nu=n+1}^{\infty}\left|\frac{\partial q_\nu^*}{\partial z^i}\right|^2 = (n-1)\sum_{i=1}^{n}|z^i|^2 + \sum_{i=1}^{n}\sum_{\nu=n+1}^{\infty}\left|\frac{\partial q_\nu^*}{\partial z^i}\right|^2,$$

where q_ν^* is the quadratic part of q_ν. Equating both members, we have

$$\sum_{i=1}^{n}\sum_{\nu=n+1}^{\infty}\left|\frac{\partial q_r^*}{\partial z^i}\right|^2 = (n+2)\sum_{i=1}^{n}|z^i|^2. \tag{9.2}$$

Now we study the value of the Riemannian curvature at the origin. First let us find the value of N at the origin. It is evidently

$$N = (I^{(n)}, 0).$$

Therefore

$$I - \bar{N}'(N\bar{N}')^{-1}N = \begin{pmatrix} 0^{(n)} & 0 \\ 0 & I^{(\infty)} \end{pmatrix}.$$

At $z = 0$, we have

$$\begin{aligned} R^* &= -\frac{\displaystyle\sum_{\nu=n+1}^{\infty}|\varphi_0 dd\varphi_\nu - \varphi_\nu dd\varphi_0|^2}{(dz\,\overline{dz'})^2} \\ &= -\sum_{\nu=n+1}^{\infty}|d\,dq_\nu^*|^2/(dz\,\overline{dz'})^2. \end{aligned} \tag{9.3}$$

Let

$$q_\nu^* = \sum_{i,j=1}^{n} b_{ij}^\nu z^i z^j, \quad b_{ij}^\nu = b_{ji}^\nu.$$

Then (9.2) becomes

$$4\sum_{i=1}^{n}\sum_{\nu=n+1}^{\infty}\left|\sum_{j=1}^{n} b_{ij}^\nu z^j\right|^2 = (n+2)\sum_{i=1}^{n}|z^i|^2.$$

Consequently

$$\sum_{\nu=n+1}^{\infty}\sum_{i=1}^{n}\left|\sum_{j=1}^{n} b_{ij}^\nu dz^j\right|^2 = \frac{(n+2)}{4}\sum_{i=1}^{n}|dz^i|^2.$$

By Schwarz' inequality, we have

$$\begin{aligned} \sum_{\nu=n+1}^{\infty}|ddq_\nu^*|^2 &= 4\sum_{\nu=n+1}^{\infty}\left|\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^\nu dz^i dz^j\right|^2 \\ &\leqslant 4\sum_{\nu=n+1}^{\infty}\left(\sum_{i=1}^{n}\left|\sum_{j=1}^{n} b_{ij}^\nu dz^j\right|^2 \cdot \sum_{i=1}^{n}|dz^i|^2\right) \\ &= (n+2)\left(\sum_{i=1}^{n}|dz^i|^2\right)^2. \end{aligned}$$

Substituting into (9.3), we have

$$-R^* \leqslant (n+2).$$

The theorem is now proved.

Remark It can be shown (Hua[1])that if the linear closure of $\{\varphi_\nu\}$ contains

$$z^i z^j, \quad 1 \leqslant i, j \leqslant n.$$

then we have the strict inequality

$$R > -n.$$

Part III

§10. Domain with constant curvature

Theorem 10 *Let $\mathfrak{D}$ be a bounded non-continuable domain with constant curvature. There is an analytic mapping carrying $\mathfrak{D}$ onto the unit sphere*

$$|z^1|^2 + \cdots + |z^n|^2 < 1.$$

Proof (1) The kernel function of a domain with constant curvature can be expressed as

$$|\xi(z)|^2 \left(1 + a \sum_{\nu=1}^{n} |\varphi_\nu(z)|^2\right)^b \qquad (a, b \text{ real}) \tag{10.1}$$

(Bochner[1]). By theorem 4, the Jacobian never vanishes in $\mathfrak{D}$. We introduce new variables

$$w^i = \varphi_i(z), \quad i = 1, \cdots, n. \tag{10.2}$$

Now we may assume that

$$|\xi(z)|^2 (1 + az\bar{z}')^b \tag{10.3}$$

is our kernel function, where $z\bar{z}' = \sum_{i=1}^{n} |z^i|^2$. Notice that $\xi(z)$ may not be that of $\xi(z)$ as before. From now on $\xi(z)$ may be not the same at each occurrence.

(2) Expand (10.3) in the following way

$$|\xi(z)|^2 \left(1 + abz\bar{z}' + \frac{1}{2}a^2 b(b-1)(z\bar{z}')^2 + \cdots\right),$$

and

$$(z\bar{z}')^m = \sum_{l_1+\cdots+l_n=m} \frac{m!}{l_1!\cdots l_n!}|z^1|^{2l_1}\cdots|z^n|^{2l_n}.$$

Let

$$\psi_l(z) = \psi_{l_1,\cdots,l_n}(z) = \sqrt{\frac{m!}{l_1!\cdots l_n!}}(z^1)^{l_1}\cdots(z^n)^{l_n}.$$

Then (10.3) can be written as

$$|\xi(z)|^2\left(1+ab(|z^1|^2+\cdots+|z^n|^2)+\cdots+a^m\frac{b(b-1)\cdots(b-m+1)}{m!}\sum_l|\psi_l(z)|^2+\cdots\right). \tag{10.4}$$

(3) Suppose that $\varphi_0(z)$, $\varphi_1(z),\cdots$ be the orthonormal system of the domain $\mathfrak{D}$, then we have

$$\sum_{\nu=0}^{\infty}\left|\frac{\varphi_\nu(z)}{\xi(z)}\right|^2 =1+ab(|z^1|^2+\cdots+|z^n|^2)+\cdots$$
$$+a^m\frac{b(b-1)\cdots(b-m+1)}{m!}\sum|\psi_l(z)|^2+\cdots. \tag{10.5}$$

Since $\xi(z)$ never vanishes in $\mathfrak{D}$, the both sided of (10.5) converge uniformly in a neighborhood of $z=0$. Applying the operator $\partial_1^m\partial_{\bar{1}}^m$ and putting $z=0$, we deduce that

$$a^m b(b-1)\cdots(b-m+1)>0$$

for all values of m. Consequently

$$a<0, \quad b<0. \tag{10.6}$$

Performing a suitable transformation if necessary, we may without loss of generality, assume that $a=-1, b=-\lambda, \lambda>0$.

(4) From

$$K(z,\bar{z}) = |\xi(z)|^2(1-z\bar{z}')^{-\lambda}$$

we have

$$K(z,\bar{w})=\xi(z)\overline{\xi(w)}(1-z\bar{w}')^{-\lambda}$$
$$=\xi(z)\xi(w)\left(1+\lambda z\bar{w}'+\cdots+\frac{\lambda(\lambda+1)\cdots(\lambda+m-1)}{m!}\sum_l\psi_l(z)\overline{\psi_l(w)}+\cdots\right), \tag{10.7}$$

which converges uniformly in w as z sufficiently small. Multiplying both sides by

$$\xi(w)\psi_{p_1,\cdots,p_n}(w)$$

and integrating with respect to w over $\mathfrak{D}$, we have then

$$\xi(z)\psi_{p_1,\cdots,p_n}(z) = \int_{\mathfrak{D}} K(z,\bar{w})\xi(w)\psi_{p_1,\cdots,p_n}(w)\dot{w}$$
$$=\xi(z)\sum_{m=0}^{\infty}\sum_{l_1+\cdots+l_n=m} a_{l_1,\cdots,l_n}\psi_{l_1,\cdots,l_n}(z), \tag{10.8}$$

where

$$a_{l_1,\cdots,l_n} = \frac{\lambda(\lambda+1)\cdots(\lambda+m-1)}{m!}\int_{\mathfrak{D}} |\xi(w)|^2\overline{\psi_{l_1,\cdots,l_n}(w)}\psi_{p_1,\cdots,p_n}(w)\dot{w}. \tag{10.9}$$

Therefore

$$\psi_{p_1,\cdots,p_n}(z) = \sum_{m=0}^{\infty}\sum_{l_1+\cdots+l_n=m} a_{l_1,\cdots,l_n}\psi_{l_1,\cdots,l_n}(z).$$

From the uniqueness of power series, we have

$$a_{l_1,\cdots,l_n} = \begin{cases} 0, & \text{if} \quad (l_1,\cdots,l_n) \neq (p_1,\cdots,p_n), \\ 1, & \text{if} \quad l_k = p_k \quad (1 \leqslant k \leqslant n). \end{cases} \tag{10.10}$$

Consequently,

$$\xi(z)\Phi_{l_1,\cdots,l_n}(z) = \sqrt{\frac{\lambda(\lambda+1)\cdots(\lambda+m-1)}{m!}}\xi(z)\psi_{l_1,\cdots,l_n}(z) \tag{10.11}$$

form a complete orthonormal system of the domain $\mathfrak{D}$.

(5) By some calculation, we find that the curvature of the domain considered is equal to $-2/\lambda$.

(6) A function belonging to $\mathfrak{L}^2$ has the expansion

$$f(z) = \xi(z)\sum_{l} a^{l_1 \quad ,l_n}\Phi_{l_1,\cdots,l_n}(z),$$
$$\sum_{l} |a^{l_1,\cdots,l_n}|^2 < \infty. \tag{10.12}$$

The power series

$$f_1(z) = \sum_{l} a^{l_1,\cdots,l_n}\Phi_{l_1,\cdots,l_n}(z) \tag{10.13}$$

form a set $\mathfrak{M}^2$. Corresponding to each $f(z)$ of $\mathfrak{L}^2$, we have a function $f_1(z)$ in $\mathfrak{M}^2$ and conversely. Since $\mathfrak{L}^2$ contains 1, therefore $\mathfrak{M}^2$ contains the function $\dfrac{1}{\xi(z)}$.

Now let us consider the coefficients satisfying $l_1+\cdots+l_n = m$. If

$$z^i = \sum_{j=1}^{n} a^i_j w^j, \tag{10.14}$$

then we have

$$\psi_{l_1,\cdots,l_n}(z) = \sum_k A_{l_1,\cdots,l_n}^{k_1,\cdots,k_n} \psi_{k_1,\cdots,k_n}(w).$$

From the relation

$$\sum_l a^l \psi_l(z) = \sum_k b^k \psi_k(w),$$

we deduce

$$a^{l_1,\cdots,l_n} = \sum_k B_{k_1,\cdots,k_n}^{l_1,\cdots,l_n} b^{k_1,\cdots,k_n}.$$

It is well known that if (a_j^i) is unitary, so are (A_l^k) and (B_k^l). Therefore we have

$$\sum_{l_1+\cdots+l_n=m} |b^{l_1,\cdots,l_n}|^2 = \sum_{l_1+\cdots+l_n=m} |a^{l_1,\cdots,l_n}|^2. \tag{10.15}$$

Therefore, we have established that, for unitary (10.14), if $f_1(z)$ belongs to $\mathfrak{M}^2$ so does $f_1(w)$. Therefore $\xi(z)f_1(w)$ belongs to $\mathfrak{L}^2$. The corresponding series converges uniformly in any region interior to $\mathfrak{D}$.

Let r be the distance from origin to the farthest point on the boundary of $\mathfrak{D}$. From the previous consideration, $f_1(z)$ is analytic every-where in the sphere $z\bar{z}' < r^2$, in particular, $\dfrac{1}{\xi(z)}$ is analytic in the sphere. Therefore, every function which is analytic in $\mathfrak{D}$, is continuable over the sphere $z\bar{z}' < r^2$. The theorem is now proved.

References

BERGMANN, S.[1]. Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques. Paris: Gauthier–Villiars, 1947.

BOCHNER, S.[1]. Curvature in Hermitian metric. *Bull. Amer. math. Soc*, 1947, 53: 179-195.

FUCHS, B.A.[1]. Über geodätische Mannigfaltigkeiten einer bei pseudokonformen Abbildungen invarianten Riemannschen Geometrie. *Recueil Math*, 1937, 2: 567-594.

HUA, L.K.[1]. On the Riemann curvature of the non-euclidean space of several complex variables. *Acta math. Sinica*, 1954, 4: 141-168.

[2] On non-continuable domain with constant curvature. *Acta Math. Sinica*, 1954, 4.

THEORY OF HARMONIC FUNCTIONS IN CLASSICAL DOMAINS*

L.K.Hua(华罗庚)And K.H. Look(陆启铿)

(*Institute of Mathematics, Academia Sinica*)

TABLE OF CONTENTS

* Received July 3, 1959. Reprinted from *Scientia Sinica*, 1959, VIII(10).

INTRODUCTION

This paper contains the results presented in several notes (Hua[3 – 5]; Hua and Look[1 – 6]; Look[2]), which were or will be published in Chinese, or were sketched in a short English summary.

§1. Heuristic statement

Let $\mathfrak{M}$ be an m-dimensional Riemannian manifold of class C^1 with the fundamental tensor $g_{ij}(i,j = 1,2,\cdots,m)$. Let $x = (x^1,\cdots,x^m)$ be an arbitrary local coordinate system. Then we have the Beltrami operator

$$\Delta = \sum_{i,j=1}^{m} g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \sum_{k=1}^{m}\left\{ \begin{array}{c} k \\ ij \end{array} \right\}\frac{\partial}{\partial x^k}\right),$$

where g^{ij} as usual denotes the contravariant tensor of g_{ij} and

$$\left\{ \begin{array}{c} k \\ ij \end{array} \right\} = \frac{1}{2}\sum_{l=1}^{m} g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{il}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

denotes the Christoffel symbol.

A real-valued function $u(x)$ of class C^2 defined in $\mathfrak{M}$ is said to be harmonic in $\mathfrak{M}$ if

$$\Delta u(x) = 0. \tag{1}$$

Now we assume that $\mathfrak{M}$ is not compact, and the matrix (g^{ij}) is positive definite in $\mathfrak{M}$ and it becomes semi-definite on the "boundary". Alternatively, it can be described with regard to the study of harmonic functions in the "whole" Riemannian manifold. Naturally it is related to the theory of linear partial differential equations of second order of elliptic type which degenerates on the "boundary" of the manifold.

Different from the case with two variables, the geometrical structure of the boundary plays an important role. According to the authors' limited knowledge, such an attempt on the theory of differential equations of degenerated elliptic type has not been achieved before. However, we do not aim at the general theory of differential equations of degenerated elliptic type, since there still remain difficulties to be overcome.

The aim of this paper is to treat those cases which are related to the theory of functions of several complex variables. More precisely, we are going to establish the theory of harmonic functions of classical domains.

According to the rank of (g^{ij}) on the boundary, we introduce the concept of slit space. The different ranks of the matrix (g^{ij}) on the "closure" of $\mathfrak{M}$ are denoted by

$$m = m_1 > m_2 > \cdots > m \geqslant 0.$$

The set of points with rank m_r is denoted by $\mathfrak{C}^{(m_r)}$. Each slit space excluding its slit is denoted by $\mathfrak{C}^{(m_r)} (r = 1, \cdots, s-1)$. The cases with which we are concerned are that $\mathfrak{C}^{(m_r)}$ is homeomorphic to the topological product of the space X and Y where X is a manifold of class C^2 and Y is a compact set.

The Dirichlet problem is defined in the following: Given a function continuous on $\mathfrak{C} = \mathfrak{C}^{(m_s)}$, whether there exists a unique solution of (1) on the closure of $\mathfrak{M}$ (certainly in the heuristic sense).

The uniqueness is established under comparatively general condition. To establish the existence theorem, our treatment is based on the Poisson integral which depends on the transibility of the space.

Let $B(x, \xi)$ be the functional determinant on $\mathfrak{C}$ of such a transformation of the group of motion of $\mathfrak{M}$ that carries the point x into a fixed point 0. Then

$$P(x, \xi) = C|B(x, \xi)|$$

with a well-determined constant C is our "Poisson" kernel. The solution of our Dirichlet problem can be expressed by

$$u(x) = \int_{\mathfrak{C}} \varphi(\xi) P(x, \xi) \dot{\xi},$$

where $\varphi(\xi)$ is the given boundary value on $\mathfrak{C}$ and $\dot{\xi}$ the volume element of $\mathfrak{C}$.

§2. Harmonic functions in the hyperbolic spaces of matrices

Let $\mathfrak{R}_{\mathrm{I}}$(or $\mathfrak{R}_{\mathrm{I}}(m, n)$) denote the domain formed by $m \times n$ complex matrices Z making the Hermitian matrix $I - Z\overline{Z'}$ positive definite, and let $\mathfrak{C}_{\mathrm{I}}$ denote the set formed by the $m \times n$ matrices U satisfying $U\overline{U'} = I$. The differential equation for this region is given by

$$\sum_{\alpha,\beta=1}^{n} \sum_{j,k=1}^{m} \left(\delta_{\alpha\beta} - \sum_{l=1}^{m} \bar{z}_{l\alpha} z_{l\beta} \right) \left(\delta_{jk} - \sum_{\gamma=1}^{m} \bar{z}_{j\gamma} z_{k\gamma} \right) \frac{\partial^2 u(Z)}{\partial \bar{z}_{j\alpha} \partial z_{k\beta}} = 0.$$

A function is said to be harmonic on the closure $\overline{\mathfrak{R}}_{\mathrm{I}}$ of $\mathfrak{R}_{\mathrm{I}}$, if it is continuous on $\overline{\mathfrak{R}}_{\mathrm{I}}$, and it satisfies, in the sense explained in the text, the equation on $\overline{\mathfrak{R}}_{\mathrm{I}} - \mathfrak{C}_{\mathrm{I}}$.

Given a continuous function $\varphi(U)$ on $\mathfrak{C}_{\mathrm{I}}$, then the Poisson integral

$$u(Z) = \frac{1}{\prod_{l=1}^{m} \omega_{2(n-l)+1}} \int_{\mathfrak{C}_{\mathrm{I}}} \varphi(U) \frac{\det(I - Z\overline{Z'})^n}{|\det(I - Z\overline{U'})|^{2n}} \dot{U}, \omega_{p-1} = \frac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$$

gives the unique solution of the Dirichlet problem.

Similar results are obtained for the hyperbolic spaces of symmetric and skew symmetric matrices.

The case $m = n$ for $\mathfrak{R}_{\rm I}$ has been considered by J. Mitchell[1], but her treatment was unsatisfactory, since the given boundary value was of a very special class of function and the uniqueness was not proved.

§3. Harmonic functions in the hyperbolic space of Lie-sphere

Let $\mathfrak{R}_{\rm IV}$ denote the domain formed by n complex variables $z = (z_1, \cdots, z_n)$ satisfying

$$1 + |zz'|^2 - 2\bar{z}z' > 0, \quad |zz'| < 1,$$

and $\mathfrak{C}_{\rm IV}$ denote the set of the vectors

$$\xi = e^{i\theta}x,$$

where x is a real n-vector satisfying $xx' = 1$.

The differential equation of the domain is given by

$$(1 + |zz'|^2 - 2z\bar{z}')\left(\sum_{\alpha=1}^{n} \frac{\partial^2 u(z)}{\partial z_\alpha \partial \bar{z}_\alpha} - 2\sum_{\alpha,\beta=1}^{n} z_\alpha \bar{z}_\beta \frac{\partial^2 u(z)}{\partial z_\alpha \partial \bar{z}_\beta}\right)$$

$$+ 2\sum_{\alpha,\beta=1}^{n} (\bar{z}_\alpha - \overline{zz'}z_\alpha)(z_\beta - zz'\bar{z}_\beta)\frac{\partial^2 u(z)}{\partial z_\alpha \partial \bar{z}_\beta} = 0. \tag{1}$$

Similarly we define a function to be harmonic on the closure of $\mathfrak{R}_{\rm IV}$.

The Dirichlet problem is also solved completely. Given a continuous function $\varphi(\xi)$ on $\mathfrak{C}_{\rm IV}$, the Poisson integral

$$u(z) = \frac{2}{\omega_1 \omega_{n-1}} \int_{\mathfrak{C}_{\rm IV}} \varphi(\xi) \frac{(1 + |zz'|^2 - 2\bar{z}z')^{n/2}}{|1 + zz'\overline{\xi\xi'} - 2\bar{\xi}z'|^n} \dot{\xi}$$

gives the unique solution of (1).

The result was published in a note (Hua and Look[2]) and a weaker result was obtained later by Lowdenslager[1]. He did not prove the uniqueness, and the path was a particular one and it approached only a part of the boundary.

§4. Applications

In Chapter IV, we shall talk briefly about some applications of the result of the present paper.

First, we obtain a convergence theorem on the theory of harmonic analysis on a unitary group. More precisely, we introduced the Abel summability, and proved that

any continuous function on the unitary group can be expressed as the Abel sum of its Fourier expansion.

Furthermore, to each continuous function on $\mathfrak{C}_{\mathrm{I}}$ we have a harmonic function in $\mathfrak{R}_{\mathrm{I}}$, but the converse is not true. That is, there may exist a harmonic function in $\mathfrak{R}_{\mathrm{I}}$, but the boundary values on $\mathfrak{C}_{\mathrm{I}}$ may not exist. We define a generalized function on $\mathfrak{C}_{\mathrm{I}}$ by means of a harmonic function on $\mathfrak{R}_{\mathrm{I}}$. According to L. Schwartz, it can also be called "a distribution".

The theory of distributions on the unitary group can be established in this way.

Finally, we give an example to illustrate the possibility to extend our results to the theory of harmonic functions with real variables.

§5. Supplementary remarks

It is interesting to remark that though we start with a differential equation, yet the harmonic function so obtained satisfies a system of differential equations of second order.

Next the harmonic property we defined is invariant under pseudoconformal mappings. A similar problem was considered before by S. Bergmann[1], but his extended classes may be altered by a pseudoconformal mapping.

The degeneracy of the differential equation on the boundary suggests a study of the differential equation of mixed type, which will not be considered in this paper.

As the classical theory of harmonic functions was generalized successfully in certain sense by Hodge[1] to n-dimensional compact Riemannian manifold, it seems to be worth while to make an attempt to consider the open manifold. Though it is far from a complete general theory, yet through these concrete examples, we see the light to proceed forward. We shall also mention the results of Duff and Spencer[1], in which they considered the open subdomain of a Riemannian manifold, yet their differential equations are positive definite throughout the domain and also on its boundary.

Chapter I

Some Preliminary Theorems

§1.1. Spaces with slits

Definition A compact metric space R is called *a slit space* or *a space with slit* S if S is a non-empty closed subset of R of which each point is an accumulating

point of $R-S$ and if $R-S$ is homeomorphic to a topological product $X\times Y$, where X is a connected m-dimensional differential manifold of class C^2, called the base space, and Y is a *compact set*, called the side space, and the homeomorphic mapping $\varphi: X\times Y\to R-S$ is called th *coordinate function*.

Consequently, X cannot be compact, for otherwise so also will be $X\times Y$, which could not be homeomorphic to $R-S$.

Example 1 The closure R of any bounded domain in an n-dimensional Euclidean space E^n can be considered as a slit space with the boundary as its slit. Now Y is a set of a single point.

Example 2 The closed bi-cylinder R:

$$|z_1|\leqslant 1,\quad |z_2|\leqslant 1$$

can be considered as a space with the boundary B as slit. Moreover, the boundary B

$$|z_1|\leqslant 1,\quad |z_2|=1\quad \text{and}\quad |z_1|=1,\quad |z_2|\leqslant 1$$

can be also considered as a space with slit C:

$$|z_1|=|z_2|=1.$$

In fact, the set $B-C$ is formed by two sets

$$|z_1|=1,\quad |z_2|<1$$

and

$$|z_1|<1,\quad |z_2|=1.$$

Each one is a product to the interior of a unit circle and a circumference of a unit circle. Therefore $B-C$ is homeomorphic to $X\times Y$ where X is the interior of a unit circle and Y the set of two circumferences.

Example 3 The space to be denoted by $\mathfrak{R}_{\rm IV}$ is formed by the complex n-vectors z satisfying

$$1+|zz'|^2-2z\bar{z}'>0,\quad 1-|zz'|>0. \tag{1.1.1}$$

The closure of the space is denoted by $\bar{\mathfrak{R}}_{\rm IV}$. Each complex vector z can be expressed as

$$z=(z_1,z_2,0,\cdots,0)\Gamma \tag{1.1.2}$$

where Γ is a real orthogonal matrix. Substituting into (1.1.1), we have

$$1 + |z_1^2 + z_2^2|^2 - 2(|z_1|^2 + |z_2|^2) > 0, \quad 1 - |z_1^2 + z_2^2| > 0.$$

Let

$$w_1 = z_1 - iz_2, \quad w_2 = z_1 + iz_2,$$

It follows that

$$(1 - |w_1|^2)(1 - |w_2|^2) > 0, \quad 1 - |w_1 w_2| > 0.$$

Consequently, we have

$$|w_1| < 1, \quad |w_2| < 1. \tag{1.1.3}$$

This is a bi-cylinder. The boundary $\mathfrak{B}_{\mathrm{IV}}$ of $\mathfrak{R}_{\mathrm{IV}}$ is therefore formed by the sets (1.1.2) with

$$|w_1| = 1, |w_2| \leqslant 1 \quad \text{and} \quad |w_1| \leqslant 1, |w_2| = 1.$$

We use $\mathfrak{C}_{\mathrm{IV}}$ to denote the set of points (1.1.2) with $|w_1| = |w_2| = 1$. We are going to prove that $\mathfrak{C}_{\mathrm{IV}}$ is a slit of $\mathfrak{B}_{\mathrm{IV}}$. More precisely, $\mathfrak{B}_{\mathrm{IV}} - \mathfrak{C}_{\mathrm{IV}}$ is homeomorphic to $X \times Y$ where X denotes the interior of a unit circle and Y denotes the cosets of $O(n)$ by $O(n-2)$, where $O(n)$ is the real orthogonal group of order n.

In fact, each point z of $\mathfrak{B}_{\mathrm{IV}} - \mathfrak{C}_{\mathrm{IV}}$ can be expressed as

$$e^{i\theta}(z_1, z_2, 0, \cdots, 0)\Gamma$$

with $|w_1| < 1$ and $w_2 = 1$ or

$$z = e^{i\theta}\left(\frac{1}{2}(1 + w_1), -\frac{i}{2}(1 - w_1), 0, \cdots, 0\right)\Gamma, \quad |w_1| < 1. \tag{1.1.4}$$

Further, since

$$\begin{aligned}
&e^{i\theta}\left(\frac{1}{2}(1 + w_1), -\frac{i}{2}(1 - w_1)\right) \\
&= \left(\frac{1}{2}(e^{i\theta} + e^{-i\theta}e^{2i\theta}w_1), -\frac{i}{2}(e^{i\theta} - e^{-i\theta}e^{2i\theta}w_1)\right) \\
&= \left(\frac{1}{2}(1 + e^{2i\theta}w_1), -\frac{i}{2}(1 - e^{2i\theta}w_1)\right)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},
\end{aligned}$$

each z of $\mathfrak{B}_{\mathrm{IV}} - \mathfrak{C}_{\mathrm{IV}}$ can be expressed as

$$z = \left(\frac{1}{2}(1 + w), -\frac{i}{2}(1 - w), 0, \cdots, 0\right)\Gamma, \quad |w| < 1. \tag{1.1.5}$$

If we have another expression

$$z = \left(\frac{1}{2}(1+w_0), -\frac{i}{2}(1-w_0), 0, \cdots, 0\right)\Gamma_0,$$

then we have, with $\Gamma_1 = \Gamma\Gamma_0^{-1}$,

$$\begin{aligned}&\left(\frac{1}{2}(1+w), -\frac{i}{2}(1-w), 0, \cdots, 0\right)\Gamma_1\\ &= \left(\frac{1}{2}(1+w_0), -\frac{i}{2}(1-w_0), 0, \cdots, 0\right).\end{aligned}$$

It follows that

$$\Gamma_1 = \begin{pmatrix} \gamma & 0 \\ 0 & \Gamma^{(n-2)} \end{pmatrix},$$

where $\Gamma^{(n-2)}$ is an $(n-2)$-rowed real orthogonal matrix and

$$\gamma = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}, \quad \text{or} \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}.$$

For the first expression of γ, from

$$\begin{aligned}&\left(\frac{1}{2}(1+w), -\frac{i}{2}(1-w)\right)\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}\\ &= \left(\frac{1}{2}(1+w_0), -\frac{i}{2}(1-w_0)\right).\end{aligned}$$

we have immediately $\psi = 0$ and the second expression is impossible.

§1.2. Partial differential equation on slit space

Let $(x^1, \cdots, x^m)$ be a local coordinate system of X, and let φ denote the mapping from $X \times Y$ onto $R - S$. A continuous function defined on $R - S$ is said to possess continuous second derivatives with respect to the coordinate of X, if $f(\varphi(x, y))$ $(x \in X, y \in Y)$ has derivatives up to the second order with respect to $x^1, \cdots, x^m$ and these derivatives are continuous on $X \times Y$.

A positive definite operator defined on X is a differential operator

$$\Delta = \sum_{j,k=1}^{m} a_{jk} \frac{\partial^2}{\partial x^j \partial x^k} + \sum_{l=1}^{m} b_l \frac{\partial}{\partial x^l}$$

with continuous coefficients a_{jk} and b_l in X such that (a_{jk}) is a positive definite symmetric matrix and Δ is independent of the choice of the local coordinates.

Theorem 1.2.1 *Let f be a function continuous in R and with the continuous second derivatives on $R - S$ with respect to the coordinate of X. If, for each $y \in Y$, f satisfies*

$$\Delta f(\varphi(x, y)) = 0,$$

then f reaches its minimal and maximal values on the slit S.

Proof Since R is a compact metric space, $f(P)$ reaches its maximal value at a point P_0 of R, that is,

$$f(P) \leqslant f(P_0).$$

for any P of R. If $P_0 \in S$, there is nothing to prove. If $P_0 \notin S$, we put $P_0 = \varphi(x_0, y_0)$ where $x_0 \in X$, $y_0 \in Y$. We define $\psi(x, y) = f(\varphi(x, y))$. Then we have $\psi(x, y) \leqslant \psi(x_0, y_0)$; in particular

$$\psi(x, y_0) \leqslant \psi(x_0, y_0).$$

In case $\psi(x, y_0) = \psi(x_0, y_0)$ for all points $x \in X$, since X is not compact, there is at least a sequence of points $x_1, \cdots, x_k, \cdots$ in X which has no accumulating point in X. But, since R is compat, the image points $P_k = \varphi(x_k, y_0)$ must have at least an accumulating point $Q \in R$, Q must lie on S. Without loss of generality, we can assume that P_k converges to Q and then $\lim f(P_k) = f(Q) = f(P_0)$.

In case $\psi(x, y_0) = \psi(x_0, y_0)$ not for all points $x \in X$, since X is connected, there is at least a point $x_1 \in X$ such that $\psi(x_1, y_0) = \psi(x_0, y_0)$, but in any neighbourhood of x_1 in X, there is at least a point $x \in X$ such that

$$\psi(x, y_0) < \psi(x_1, x_0).$$

We choose a connected neighbourhood $U(x_1)$ of x_1 in X, which is contained in a local coordinate neighbourhood of X and the closure of which is compact. Since

$$\Delta\psi(x, y_0) = 0,$$

we can apply the extremal principle of linear partial differential equations of elliptic type (see, for example, Miranda [1], p. 5) to the domain $U(x_1)$ and see that this case is impossible.

Consequently, we always have $f(P) \leqslant \max\limits_{Q \in S}(Q)$. By a similar method, we also have $f(P) \geqslant \min\limits_{Q \in S} f(Q)$. The proof of the theorem is complete.

A sequence of spaces

$$R_1 \supset R_2 \supset \cdots \supset R_k$$

is called a chain of slit spaces, if each R_ν is a slit space with $R_{\nu+1}$ as its slit ($\nu = 1, \cdots, k-1$).

Theorem 1.2.2 *Denote the slit of R_k by $\mathfrak{C}$. Let Δ_ν be a positive definite operator defined in X_ν, the base space of R_ν. Suppose that f is a continuous function defined in R_1 and on each $R_\nu - R_{\nu+1}$ ($\nu = 1, \cdots, k$; $R_{k+1} = \mathfrak{C}$), it possesses continuous second derivatives with respect to the coordinate of X_ν. If for each point y of Y_ν, the side space of R_ν, f satisfies*

$$\Delta_\nu f(\varphi_\nu(x, y)) = 0, \quad (\nu = 1, \cdots, k)$$

for x on X_ν, where φ_ν is the coordinate function of R_ν, then f reaches its maximal and minimal values on $\mathfrak{C}$.

The theorem is an immediate consequence by successive applications of our previous theorem.

§1.3. Several lemmas

Theorem 1.3.1 *Let m be an integer $\geqslant 2$ and $0 < r < 1$. Then we have*

$$\int \cdots \int_{x_1^2+\cdots+x_m^2=1} \frac{(1-r^2)^{m/2}}{|1 - r(x_1 - ix_2)|^m} \dot{x} = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)},$$

where $\dot{x}$ denotes the volume element of the sphere $x_1^2 + \cdots + x_m^2 = 1$.

Proof Changing variables

$$x_1 = \rho\ \cos\theta, x_2 = \rho\ \sin\theta, x_3 = x_3, \cdots, x_m = x_m,$$

where $\rho = \sqrt{1 - x_3^2 - \cdots - x_m^2}$, we have

$$\begin{aligned} dx_1^2 + \cdots + dx_m^2 &= \rho^2 d\theta^2 + d\rho^2 + \sum_{\nu=3}^{m} dx_\nu^2 \\ &= \rho^2 d\theta + \frac{1}{\rho^2}(x_3 dx_3 + \cdots + x_m dx_m)^2 + \sum_{\nu=3}^{m} dx_\nu^2. \end{aligned}$$

Since

$$\begin{aligned} &\det\left(I^{(m-2)} + \frac{1}{\rho^2}(x_3, \cdots, x_m)'(x_3, \cdots, x_m)\right) \\ &= 1 + \frac{1}{\rho^2}(x_3^2 + \cdots + x_m^2) = \frac{1}{\rho^2}, \end{aligned}$$

we have

$$\dot{x} = d\theta dx_3 \cdots dx_m,$$

and

$$\int_{x_1^2+\cdots+x_m^2=1}\cdots\int \frac{\dot{x}}{|1-r(x_1-ix_2)|^m} = \int_{x_3^2+\cdots+x_m^2<1}\cdots\int dx_3\cdots dx_m \int_0^{2\pi}\frac{d\theta}{|1-r\rho e^{-i\theta}|^m}.$$

Using the development

$$(1-r\rho e^{-i\theta})^{-\frac{m}{2}} = \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}m+k\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma(k+1)}(r\rho e^{-i\theta})^k,$$

we have

$$\int_0^{2\pi}\frac{d\theta}{|1-r\rho e^{-i\theta}|^m} = 2\pi\sum_{k=0}^{\infty}\left(\frac{\Gamma\left(\frac{m}{2}+k\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma(k+1)}\right)^2(r\rho)^{2k}.$$

On the other hand, since

$$\int_{x_3^2+\cdots+x_m^2<1}\cdots\int (1-x_3^2-\cdots-x_m^2)^k dx_3\cdots dx_m = \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^{m-2}\Gamma(k+1)}{\Gamma\left(\frac{m}{2}+k\right)},$$

we have consequently,

$$\begin{aligned}\int_{x_1^2+\cdots+x_m^2=1}\cdots\int \frac{\dot{x}}{|1-r(x_1-ix_2)|^m} &= \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{m}{2}+k\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma(k+1)}r^{2k}\\ &= \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}(1-r^2)^{-\frac{m}{2}}.\end{aligned}$$

Theorem 1.3.2 *if $\varphi(x_1,\cdots,x_m)$ is a continuous function on the sphere $x_1^2+\cdots+x_m^2=1$, then*

$$\lim_{r\to 1}\int_{x_1^2+\cdots+x_m^2=1}\cdots\int \varphi(x_1,\cdots,x_m)\frac{(1-r^2)^{\frac{m}{2}}}{|1-r(x_1-ix_2)|^m}\dot{x} = \frac{2\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}\varphi(1,0,\cdots,0).$$

Proof By the previous theorem we deduce that

$$\int_{x_1^2+\cdots+x_m^2=1}\cdots\int \varphi(1,0,\cdots,0)\frac{(1-r^2)^{\frac{m}{2}}}{|1-r(x_1-ix_2)|^m}\dot{x} = \omega_{m-1}\varphi(1,0,\cdots,0),$$

where $\omega_{m-1} = 2\pi^{\frac{m}{2}}/\Gamma\left(\frac{m}{2}\right)$. It is sufficient to prove that, for any given $\varepsilon > 0$, we can choose r sufficiently near to 1 such that

$$\left|\int\cdots\int_{x_1^2+\cdots+x_m^2=1} (\varphi(x_1,\cdots,x_m) - \varphi(1,0,\cdots,0))\frac{(1-r^2)^{\frac{m}{2}}}{|1-r(x_1-ix_2)|^m}\dot{x}\right| < \varepsilon. \qquad (1.2.1)$$

We use spherical coordinates

$$x_1 = \cos\theta_1, x_2 = \sin\theta_1\cos\theta_2, \cdots, x_m = \sin\theta_1\cdots\sin\theta_{m-1},$$

and notice that when $\theta_1 = 0, (x_1,\cdots,x_m) = (1,0,\cdots,0)$. It is known that

$$\dot{x} = \sin^{m-2}\theta_1\sin^{m-3}\theta_2\cdots\sin\theta_{m-2}d\theta_1\cdots d\theta_{m-1},$$

and the integral in (1.2.1) becomes

$$\begin{aligned}&\int_0^{2\pi} d\theta_{m-1}\int_0^{\pi}\sin\theta_{m-2}d\theta_{m-2}\cdots\int_0^{\pi}\sin^{m-3}\theta_2 d\theta_2\\ &\times\int_0^{\pi}(\varphi(x_1,\cdots,x_m)-\varphi(1,0,\cdots,0))\frac{(1-r^2)^{\frac{m}{2}}\sin^{m-2}\theta_1 d\theta_1}{|1-r(\cos\theta_1-i\sin\theta_1\cos\theta_2)|^m}\\ =&I_1+I_2,\end{aligned}$$

where I_1 equals the part of the integral with θ_1 integrated from 0 to δ and I_2 the remaining part of the integral.

Since $\varphi(x_1,\cdots,x_m)$ is continuous on $x_1^2+\cdots+x_m^2=1$, we can choose $\delta(>0)$ so small that, for $0 \leqslant \theta_1 \leqslant \delta$, we have

$$|\varphi(x_1,\cdots,x_m)-\varphi(1,0,\cdots,0)| < \frac{1}{2\omega_{m-1}}\varepsilon.$$

Hence

$$\begin{aligned}|I_1| <& \frac{\varepsilon}{2}\cdot\frac{1}{\omega_{m-1}}\int_0^{2\pi}d\theta_{m-1}\int_0^{\pi}\sin\theta_{m-2}d\theta_{m-2}\cdots\int_0^{\pi}\sin^{m-3}\theta_2 d\theta_2\\ &\times\int_0^{\delta}\frac{(1-r^2)^{\frac{m}{2}}\sin^{m-2}\theta_1 d\theta_1}{|1-r(\cos\theta_1-i\sin\theta_1\cos\theta_2)|^m}\\ \leqslant& \frac{\varepsilon}{2}\cdot\frac{1}{\omega_{m-1}}\int\cdots\int_{x_1^2+\cdots+x_m^2=1}\frac{(1-r^2)^{\frac{m}{2}}}{|1-r(x_1-ix_2)|^m}\dot{x} = \frac{\varepsilon}{2}.\end{aligned}$$

For a fixed δ, we have, for $\delta \leqslant \theta_1 \leqslant \pi$,

$$\begin{aligned}|1-r(\cos\theta_1-i\sin\theta_1\cos\theta_2)|^m &= |(1-r\cos\theta_1)^2+r^2\sin^2\theta_1\cos^2\theta_2|^{\frac{m}{2}}\\ &\geqslant (1-r\cos\theta_1)^m > (1-\cos\delta)^m\\ &= 2^m\sin^{2m}\frac{\delta}{2}.\end{aligned}$$

Let M be the upper bound of $|\varphi(x_1, \cdots, x_m)|$, and taking r sufficiently near to 1 such that

$$0 < (1-r^2)^{\frac{m}{2}} < \frac{2^{m-1}\sin^{2m}\dfrac{\delta}{2}}{M\omega_{m-1}} \cdot \frac{\varepsilon}{2}.$$

Then obviously

$$\begin{aligned}
|I_2| &\leqslant 2M \int_0^{2\pi} d\theta_{m-1} \int_0^{\pi} \sin\theta_{m-2} d\theta_{m-2} \cdots \int_0^{\pi} \sin^{m-3}\theta_2 d\theta_2 \\
&\quad \times \int_\delta^\pi \frac{(1-r^2)^{\frac{m}{2}} \sin^{m-2}\theta_1}{|1-r(\cos\theta_1 - i\sin\theta_1\cos\theta_2)|^m} d\theta_1 \\
&\leqslant \frac{2M(1-r^2)^{\frac{m}{2}}}{2^m \sin^{2m}\dfrac{\delta}{2}} \int_0^{2\pi} d\theta_{m-1} \int_0^{\pi} \sin\theta_{m-2} d\theta_{m-2} \cdots \int_0^{\pi} \sin^{m-3}\theta_2 d\theta_2 \\
&\quad \times \int_0^\pi \sin^{m-2}\theta_1 d\theta_1 \\
&= \frac{M\omega_{m-1}(1-r^2)^{\frac{m}{2}}}{2^{m-1}\sin^{2m}\dfrac{\delta}{2}} < \frac{\varepsilon}{2}.
\end{aligned}$$

The inequality (1.2.1) follows from the estimations of I_1 and I_2.

Chapter II

The Dirichlet Problems in the Hyperbolic Spaces of Matrices

§2.1. A differential operator of $\mathfrak{R}_{\mathrm{I}}$

We ues $\mathfrak{R}_{\mathrm{I}}$ to denote the domain formed by $m \times n$ matrices $Z = (z_{j2})_{1 \leqslant j \leqslant m, 1 \leqslant 2 \leqslant n}$ $(m \leqslant n)$ making the Hermitian matrices

$$I - Z\bar{Z}' \tag{2.1.1}$$

positive definite, where Z' and $\bar{Z}$ denote the transposed and complex conjugate matrices of Z respectively. For simplicity, we also use $H > 0$ and $H \geqslant 0$ to denote that the Hermitian matrix H is positive definite and positive semi-definite respectively.

We shall consider the linear differential operator

$$\Delta_{\mathrm{I}} = \sum_{\alpha,\beta=1}^{n} \sum_{j,k=1}^{m} \left(\delta_{\alpha\beta} - \sum_{l=1}^{m} \bar{z}_{l\alpha} z_{l\beta}\right) \left(\delta_{jk} - \sum_{\gamma=1}^{n} \bar{z}_{j\gamma} z_{k\gamma}\right) \frac{\partial^2}{\partial z_{k\beta} \partial \bar{z}_{j\alpha}}. \tag{2.1.2}$$

where $\delta_{\alpha\beta} = 0$ or 1 according as $\alpha \neq \beta$ or $\alpha = \beta$.

We introduce a matrix operator

$$\partial_Z = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{1n}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial z_{m1}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}. \tag{2.1.3}$$

Then the differential operator Δ_{I} can be expressed as

$$\Delta_{\mathrm{I}} = \mathrm{tr}((I - Z\bar{Z}')\bar{\partial}_Z(I - \bar{Z}'Z)\partial_Z'),$$

where tr M denotes the trace of a matrix M. Notice that the expression is performed as a formal multiplication; we are not applying $\bar{\partial}_Z$ as differential operator on $(I-\bar{Z}'Z)$. Such an agreement will be understood throughout this paper, i.e., if A is a matrix operator and B is any matrix, then AB means the formal product, and if u is a function, $A \circ u$ means that we apply the operator A on u.

We shall study first the covariant property of ∂_Z under the group Γ^{I} of motions of $\mathfrak{R}_{\mathrm{I}}$. It is known (Hua[1]) that the transformations of Γ^{I} can be written as

$$W = (AZ + B)(CZ + D)^{-1} = (Z\overline{B'} + \overline{A'})^{-1}(Z\overline{D'} + \overline{C'}), \tag{2.1.4}$$

where $A = A^{(m)}$, $B = B^{(m,n)}$, $C = C^{(n,m)}$ and $D = D^{(n)}$ satisfy the relations:

$$A\overline{A'} - B\overline{B'} = I, \quad A\overline{C'} = B\overline{D'}, \quad C\overline{C'} - D\overline{D'} = -I, \tag{2.1.5}$$

or what is the same thing,

$$\overline{A'}A - \overline{C'}C = I, \quad \overline{A'}B = \overline{C'}D, \quad \overline{B'}B - \overline{D'}D = -I. \tag{2.1.6}$$

By differentiating (2.1.4), we have

$$\begin{aligned} dW &= [AdZ - (AZ + B)(CZ + D)^{-1}CdZ](CZ + D)^{-1} \\ &= (Z\overline{B'} + \overline{A'})^{-1}[(Z\overline{B'} + \overline{A'})A - (Z\overline{D'} + \overline{C'})C]dZ(CZ + D)^{-1} \\ &= (Z\overline{B'} + \overline{A'})^{-1}dZ(CZ + D)^{-1}. \end{aligned} \tag{2.1.7}$$

From the relations

$$\frac{\partial}{\partial z_{j\alpha}} = \sum_{k=1}^{m}\sum_{\beta=1}^{n} \frac{\partial w_{k\beta}}{\partial z_{j\alpha}} \frac{\partial}{\partial w_{k\beta}}, \quad dw_{j\alpha} = \sum_{k=1}^{m}\sum_{\beta=1}^{n} \frac{\partial w_{j\alpha}}{\partial z_{k\beta}} dz_{k\beta},$$

we deduce easily

$$\partial'_Z = (CZ + D)^{-1}\partial'_W(Z\overline{B'} + \overline{A'})^{-1}. \tag{2.1.8}$$

From (2.1.4) and (2.1.6), we have

$$\begin{aligned} I - \bar{W}'W =& I - \overline{(CZ+D)'}^{-1}\overline{(AZ+B)'}(AZ+B)(CZ+D)^{-1} \\ =& \overline{(CZ+D)'}^{-1}[\overline{(CZ+D)'}(CZ+D) \\ & - \overline{(AZ+B)'}(AZ+B)](CZ+D)^{-1} \\ =& (\bar{Z}'\bar{C}' + \bar{D}')^{-1}(I - \bar{Z}'Z)(CZ+D)^{-1}, \end{aligned} \tag{2.1.9}$$

and

$$I - W\overline{W'} = (Z\overline{B'} + \overline{A'})^{-1}(I - Z\overline{Z'})(B\overline{Z'} + A)^{-1}. \tag{2.1.10}$$

Combining with (2.1.8), we have

$$\begin{aligned} &(I - Z\bar{Z}')\bar{\partial}_Z(I - \bar{Z}'Z)\partial'_Z \\ =& (Z\bar{B}' + \bar{A}')(I - W\bar{W}')\bar{\partial}_W(I - \bar{W}'W)\partial'_W. \end{aligned} \tag{2.1.11}$$

Consequently we have the important identity

$$\begin{aligned} \Delta_{\mathrm{I}} &= \mathrm{tr}((I - Z\overline{Z'})\overline{\partial_Z}(I - \overline{Z'}Z)\partial'_Z) \\ &= \mathrm{tr}((I - W\overline{W'})\overline{\partial_W}(I - \overline{W'}W)\partial'_W). \end{aligned} \tag{2.1.12}$$

That is, we have

Theorem 2.1.1 *The differential operator Δ_{I} is invariant under the transformations* (2.1.4) *of the group Γ^{I}.*

§2.2. Harmonic functions in $\mathfrak{R}_{\mathrm{I}}$

A real-valued function $u(Z)$ possessing continuous second derivatives is said to be harmonic in $\mathfrak{R}_{\mathrm{I}}$, if it satisfies

$$\Delta_{\mathrm{I}} \circ u(Z) = 0 \tag{2.2.1}$$

in $\mathfrak{R}_{\mathrm{I}}$.

From Theorem 2.1.1, we deduce immediately

Theorem 2.2.1 *The property "harmonic" is invariant under the group Γ^{I}. More precisely, if $u(Z)$ is harmonic in $\mathfrak{R}_{\mathrm{I}}$, so is $u((AZ+B)(CZ+D)^{-1})$ where A, B, C and D satisfy* (2.1.5).

In $\mathfrak{R}_{\mathrm{I}}$, we have a Poisson kernel (Hua[4])

$$P_{\mathrm{I}}(Z, U) = \frac{1}{V(\mathfrak{C}_{\mathrm{I}})} \cdot \frac{\det(I - Z\overline{Z}')^n}{|\det(I - Z\overline{U}')|^{2n}}, \tag{2.2.2}$$

where Z belongs to $\mathfrak{R}_{\mathrm{I}}$ and $U(=U^{(m,n)})$ is an $m\times n$ matrix satisfying

$$U\bar{U}'=I^{(m)}, \tag{2.2.3}$$

and $\mathfrak{C}_{\mathrm{I}}$ consists of all matrices U satisfying (2.2.3). Further $V(\mathfrak{C}_{\mathrm{I}})$ denotes the total volume of $\mathfrak{C}_{\mathrm{I}}$, which is known to be①

$$V(\mathfrak{C}_{\mathrm{I}})=\frac{2^m\pi^{mn-\frac{1}{2}m(m-1)}}{(n-m)!\cdots(n-1)!}=\prod_{l=1}^{m}\frac{2\pi^{n-l+1}}{(n-l)!}=\prod_{l=1}^{m}\omega_{2(n-l)+1}, \tag{2.2.4}$$

where ω_{2p-1} is the volume of the sphere $|z_1|^2+\cdots+|z_p|^2=1$.

From (2.1.10), it follows that (2.1.4) carries the characteristic manifold onto itself. Let

$$V=(AU+B)(CU+D)^{-1}, \tag{2.2.5}$$

then we have

$$I-W\bar{V}'=(Z\bar{B}'+\bar{A}')^{-1}(I-Z\bar{U}')(B\bar{U}'+A)^{-1}, \tag{2.2.6}$$

and then

$$\frac{\det(I-W\bar{W}')^n}{|\det(I-W\bar{V}')|^{2n}}=\frac{\det(I-Z\bar{Z}')^n}{|\det(I-Z\bar{U}')|^{2n}}|\det(B\bar{U}'+A)|^{2n}. \tag{2.2.7}$$

Consequently we have the following

Theorem 2.2.2 *Under the transformation*(2.1.4), *the Poisson kernel satisfies*

$$P_{\mathrm{I}}(W,V)=P_{\mathrm{I}}(Z,U)|\det(B\bar{U}'+A)|^{2n}. \tag{2.2.8}$$

Theorem 2.2.3 *The Poisson kernel $P_{\mathrm{I}}(Z,U)$ satisfies a system of partial differential equations*

$$\bar{\partial}_Z(I-\bar{Z}'Z)\partial_Z'\circ P_{\mathrm{I}}(Z,U)=0, \tag{2.2.9}$$

or, what is the same thing,

$$\sum_{\alpha\beta=1}^{n}\left(\delta_{\alpha\beta}-\sum_{k=1}^{m}\bar{z}_{k\alpha}z_{k\beta}\right)\frac{\partial^2P_{\mathrm{I}}(Z,U)}{\partial\bar{z}_{j\alpha}\partial z_{l\beta}}=0,\quad (j,l-1,\cdots,m). \tag{2.2.10}$$

Proof (i) First we prove that at the point $Z=0, P_{\mathrm{I}}\doteq(Z,U)$ satisfies (2.2.9). We write $U=(u_{j\alpha})$. At $Z=0$, we have

$$\left[\sum_{\alpha,\beta=1}^{n}\left(\delta_{\alpha\beta}-\sum_{k=1}^{n}\bar{z}_{k\alpha}z_{k\beta}\right)\frac{\partial^2P_I(Z,U)}{\partial\bar{z}_{j\alpha}\partial z_{l\beta}}\right]_{Z=0}$$

① This volume differs from that given previously in a monograph of Hua[4], by a factor $2^{m(n-1)-\frac{1}{2}m(m-1)}$. This fact is due to the definition of the volume element $\dot{U}$.

$$
\begin{aligned}
&= \left[\sum_{\alpha=1}^{n} \frac{\partial^2 P_I(Z,U)}{\partial \bar{z}_{j\alpha} \partial z_{l\alpha}} \right]_{Z=0} \\
&= \frac{1}{V(\mathfrak{C}_{\rm I})} \left[\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial \bar{z}_{j\alpha} \partial z_{l\alpha}} \det(I - Z\bar{Z}')^n \det(I - Z\bar{U}')^{-n} \det(I - U\bar{Z}')^{-n} \right]_{Z=0} \\
&= \frac{1}{V(\mathfrak{C}_{\rm I})} \left[\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial \bar{z}_{j\alpha} \partial z_{l\alpha}} \left(1 - n \sum_{\beta=1}^{n} \sum_{k=1}^{m} |z_{k\beta}|^2 + \cdots \right) \right. \\
&\quad \left. \times \left(1 + n \sum_{\beta=1}^{n} \sum_{k=1}^{m} z_{k\beta} \bar{u}_{k\beta} + \cdots \right) \left(1 + n \sum_{\beta=1}^{n} \sum_{k=1}^{m} \bar{z}_{k\beta} u_{k\beta} + \cdots \right) \right]_{Z=0} \\
&= \frac{1}{V(\mathfrak{C}_{\rm I})} \left[\sum_{\alpha=1}^{n} (-n)\delta_{jl} + n^2 \sum_{\alpha=1}^{n} u_{j\alpha} \bar{u}_{l\alpha} \right]. \qquad (2.2.11)
\end{aligned}
$$

In matrix form, (2.2.11) can be written as

$$[\bar{\partial}_Z (I - \bar{Z}'Z) \partial'_Z \circ P_{\rm I}(Z,U)]_{Z=0} = \frac{1}{V(\mathfrak{C}_{\rm I})} [-n^2 I + n^2 U\bar{U}'] = 0,$$

since $U\bar{U}' = I$.

(ii) Let T be any point of $\mathfrak{R}_{\rm I}$, then there exist matrices A and D such that

$$(I - T\bar{T}')^{-1} = \bar{A}'A$$

and

$$(I - \bar{T}'T)^{-1} = \bar{D}'D.$$

We put

$$B = -AT, \quad C = -D\bar{T}'.$$

Obviously A, B, C, D satisfy the conditions (2.1.6) and the corresponding transformation (2.1.4) takes the form

$$W = A(Z - T)(I - \bar{T}'Z)^{-1} D^{-1}, \qquad (2.1.12)$$

which carries the point $Z = T$ into $W = 0$. According to (2.1.8), (2.1.9) and Theorem 2.2.2, we have

$$
\begin{aligned}
&[\bar{\partial}_Z (I - \bar{Z}'Z) \partial'_Z \circ P_{\rm I}(Z,U)]_{Z=T} \\
&= |\det(I - T\bar{U}')A|^{-2n} (I - T\bar{T}')^{-1} A^{-1} [\bar{\partial}_W (I - \bar{W}'W) \partial'_W \circ P_{\rm I}(W,V)]_{W=0} \\
&\quad \times \bar{A}'^{-1} (I - T\bar{T}')^{-1} = 0.
\end{aligned}
$$

The theorem is now completely proved.

As a consequence, we have

Theorem 2.2.4 *The Poisson kernel $P_{\mathrm{I}}(Z,U)$ is harmonic in $\mathfrak{R}_{\mathrm{I}}$ with respect to the variable Z.*

By the process of differentiating under integral signs we have

Theorem 2.2.5 *For any continuous function $\varphi(U)$ on $\mathfrak{C}_{\mathrm{I}}$, the Poisson integral*

$$u(Z)=\int_{\mathfrak{C}_{\mathrm{I}}}\varphi(U)P_{\mathrm{I}}(Z,U)\dot{U} \tag{2.2.13}$$

gives a function harmonic in $\mathfrak{R}_{\mathrm{I}}$.

In fact,

$$\Delta_{\mathrm{I}}\circ u(Z)=\int_{\mathfrak{C}_{\mathrm{I}}}\varphi(U)\Delta_{\mathrm{I}}\circ P_{\mathrm{I}}(Z,U)\dot{U}=0.$$

§2.3. The structure of the closure of $\mathfrak{R}_{\mathrm{I}}$

More explicitly, sometimes we use $\mathfrak{R}_{\mathrm{I}}(m,n)$ and $\mathfrak{C}_{\mathrm{I}}(m,n)$ to denote $\mathfrak{R}_{\mathrm{I}}$ and $\mathfrak{C}_{\mathrm{I}}$ respectively.

Let $\mathfrak{B}_{\mathrm{I}}^{(m-r)}$ denote the set of matrices Z such that $I-Z\bar{Z}'\geqslant 0$ and of rank $\leqslant r$. Evidently $\mathfrak{B}_{\mathrm{I}}^{(0)}$ is the closure of $\mathfrak{R}_{\mathrm{I}}$ and $\mathfrak{B}_{\mathrm{I}}^{(m)}$ is $\mathfrak{C}_{\mathrm{I}}$.

Theorem 2.3.1 *We let*

$$\mathfrak{C}_{\mathrm{I}}^{(r)}=\mathfrak{B}_{\mathrm{I}}^{(m-r)}-\mathfrak{B}_{\mathrm{I}}^{(m-r+1)},\quad (r=0,1,\cdots,m-1).$$

Then $\mathfrak{C}_{\mathrm{I}}^{(r)}$ is invariant and forms a transitive set under the transformation group Γ^{I}. More precisely, there is a transformation of the group Γ^{I} to carry any given point of $\mathfrak{C}_{\mathrm{I}}^{(r)}$ into

$$\begin{pmatrix} I^{(m-r)} & 0^{(m-r,n-m+r)} \\ 0^{(r,m-r)} & 0^{(r,n-m+r)} \end{pmatrix}.$$

Proof From (2.1.10) we have

$$(Z\bar{B}'+\bar{A}')(I-W\bar{W}')(B\bar{Z}'+A)=I-Z\bar{Z}'.$$

It follows that

$$r(I-Z\bar{Z}')\leqslant r(I-W\bar{W}'),$$

where $r(X)$ denote the rank of X. The inversion establishes

$$r(I-Z\bar{Z}')\geqslant r(I-W\bar{W}').$$

Therefore $\mathfrak{C}_{\mathrm{I}}^{(r)}$ is a set invariant under Γ^{I}.

Let Z be any point on $\mathfrak{C}_{\mathrm{I}}^{(r)}$; there are unitary matrices $U = U^{(m)}$ and $V = V^{(n)}$ such that

$$UZV = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix}, \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0.$$

Since the rank of $I - Z\bar{Z}'$ is equal to r, we have $\lambda_1 = \cdots = \lambda_{m-r} = 1$ and $\lambda_{m-r+1} < 1$. The theorem follows from the following

Theorem 2.3.2 *Let*

$$T_0 = U_0' \begin{pmatrix} 0 & 0 \\ 0 & T_1^{(r,n-m+r)} \end{pmatrix} V_0, \quad I^{(r)} - T_1\bar{T}'_1 > 0.$$

The transformation

$$W = A(Z - T_0)(I - \bar{T}'_0 Z)^{-1} D^{-1}$$

of Γ^{I}, where

$$A = \begin{pmatrix} I^{(m-r)} & 0 \\ 0 & A_1 \end{pmatrix} \bar{U}_0, \quad D = \begin{pmatrix} I & 0 \\ 0 & D_1 \end{pmatrix} V_0$$

and

$$\bar{A}'_1 A_1 = (I - T_1\bar{T}'_1)^{-1}, \quad \bar{D}'_1 D_1 = (I - \bar{T}'_1 T_1)^{-1},$$

carries $T = U_0' \begin{pmatrix} I^{(m-r)} & 0 \\ 0 & T_1 \end{pmatrix} V_0$ *into*

$$\begin{pmatrix} I^{(m-r)} & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$Z = U_0' \begin{pmatrix} \rho I^{(m-r)} & 0 \\ 0 & Z_1 \end{pmatrix} V_0, \quad 0 \leqslant \rho < 1, I - Z_1\bar{Z}'_1 > 0$$

into

$$W = U_0' \begin{pmatrix} \rho I^{(m-r)} & 0 \\ 0 & W_1 \end{pmatrix} V_0$$

with

$$W_1 = A_1(Z_1 - T_1)(I - \bar{T}'_1 Z_1)^{-1} D_1^{-1}.$$

The present theorem is evident.

Theorem 2.3.3 *The closure* $\mathfrak{B}_{\rm I}^{(0)}$ *of* $\mathfrak{R}_{\rm I}$ *has the following chain of slit spaces:*

$$\mathfrak{B}_{\rm I}^{(0)} \supset \mathfrak{B}_{\rm I}^{(1)} \supset \cdots \supset \mathfrak{B}_{\rm I}^{(m-1)}.$$

More precisely, $\mathfrak{C}_{\rm I}^{(r)} = \mathfrak{B}_{\rm I}^{(m-r)} - \mathfrak{B}_{\rm I}^{(m-r+1)}$ *(where* $r = 1, \cdots, m$ *and* $\mathfrak{B}_{\rm I}^{(m)} = \mathfrak{C}_{\rm I}$ *is the slit of* $\mathfrak{B}_{\rm I}^{(m-1)}$*) is homeomorphic to the topological product*

$$\mathfrak{R}_{\rm I}(r, n-m+r) \times \mathfrak{M}_{\rm I}^{(m-r)},$$

where $\mathfrak{M}_{\rm I}^{(s)}$ *is defined in the following way.*

We use $\mathfrak{U}(m)$ *to denote the* m*-rowed unitary group. Let* s *be an integer satisfying* $0 < s < m$*; we consider the pairs of matrices*

$$(U, V), \quad U \in \mathfrak{U}(m), \quad V \in \mathfrak{U}(n).$$

Two pairs (U, V) *and* (U_1, V_1) *are called equivalent, if there exist three unitary matrices* $U^{(s)}$, $U^{(m-s)}$ *and* $V^{(n-s)}$ *such that*

$$U = \begin{pmatrix} U^{(s)} & 0 \\ 0 & U^{(m-s)} \end{pmatrix} U_1, \quad V = \begin{pmatrix} \bar{U}^{(s)} & 0 \\ 0 & V^{(n-s)} \end{pmatrix} V_1.$$

By the equivalence, we classify pairs into classes. Each class is considered as an element; the totality of elements is defined to be the set $\mathfrak{M}_{\rm I}^{(s)}$*.*

Proof It is known that each element of $\mathfrak{C}_{\rm I}^{(r)}$ can be expressed as

$$Z = U' \begin{pmatrix} I^{(m-r)} & 0 \\ 0 & W \end{pmatrix} V,$$

where $U = U^{(m)}$ and $V = V^{(n)}$ are unitary matrices and $W = W^{(r, n-m+r)}$ satisfies

$$I^{(r)} - W\bar{W}' > 0.$$

If there is another expression

$$Z = U_1' \begin{pmatrix} I^{(m-r)} & 0 \\ 0 & W_1 \end{pmatrix} V_1,$$

we have then

$$U_2' \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & W_1 \end{pmatrix} V_2, \tag{2.3.1}$$

where

$$U_2 = UU_1^{-1}, \quad V_2 = V_1 V^{-1}.$$

From (2.3.1), we have

$$U_2'\begin{pmatrix} I & 0 \\ 0 & W\overline{W'} \end{pmatrix}\overline{U}_2 = \begin{pmatrix} I & 0 \\ 0 & W_1 \end{pmatrix} V_2\overline{V'}_2 \begin{pmatrix} I & 0 \\ 0 & W\overline{W'}_1 \end{pmatrix},$$

i.e.,

$$U_2'\begin{pmatrix} I & 0 \\ 0 & W\overline{W'} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & W_1\overline{W'}_1 \end{pmatrix} U_2'.$$

Since none of the characteristic roots of $W\bar{W}'$ and $W_1\bar{W}'_1$ is equal to 1, consequently

$$U_2 = \begin{pmatrix} U^{(m-r)} & 0 \\ 0 & U^{(r)} \end{pmatrix}.$$

By argument similar to that of (2.3.1), we deduce

$$V_2 = \begin{pmatrix} \bar{U}^{(m-r)} & 0 \\ 0 & V^{(n-m+r)} \end{pmatrix}.$$

The theorem follows.

§2.4. The boundary properties of the Poisson integral of $\mathfrak{R}_{\mathrm{I}}$

Now we study the properties of the Poisson integral

$$u(Z) = \int_{\mathfrak{C}_1} \varphi(U)P_{\mathrm{I}}(Z,U)\dot{U} \tag{2.4.1}$$

as Z approaches a boundary point of $\mathfrak{R}_{\mathrm{I}}$ from its interior.

Theorem 2.4.1 *Let $m > 1$. Let $\varphi(U)$ be a real-valued function continuous on the characteristic manifold $\mathfrak{C}_{\mathrm{I}}(m,n)$ of $\mathfrak{R}_{\mathrm{I}}(m,n)$ and let*

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & Z_0^{(m-1,n-1)} \end{pmatrix}, \quad Z = \begin{pmatrix} \rho & 0 \\ 0 & Z_0^{(m-1,n-1)} \end{pmatrix}$$

with $0 \leqslant \rho < 1$ and $I - Z_0\bar{Z}'_0 > 0$. Then

$$\lim_{\rho\to 1}\int_{\mathfrak{C}_{\mathrm{I}}(m,n)} \varphi(U)P_{\mathrm{I}}(Z,U)\dot{U}$$
$$= \int_{\mathfrak{C}_{\mathrm{I}}(m-1,n-1)} \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & U_1 \end{pmatrix}\right) P_I^{(m-1,n-1)}(Z_0,U_0)\dot{U}_0 \tag{2.4.2}$$

uniformly with respect to Z_0, where $U_0 \in \mathfrak{C}_{\mathrm{I}}(m-1,n-1)$ and $P_{\mathrm{I}}^{(m-1,n-1)}(Z_0,U_0)$ is the Poisson kernel of $\mathfrak{R}_{\mathrm{I}}(m-1,n-1)$, that is

$$P_{\mathrm{I}}^{(m-1,n-1)}(Z_0,U_0) = \frac{1}{V(\mathfrak{C}_{\mathrm{I}}(m-1,n-1))}\cdot\frac{\det(I - Z_0\bar{Z}'_0)^{n-1}}{|\det(I - Z_0\bar{U}'_0)|^{2(n-1)}}.$$

Proof First let us consider the special case with $Z_0 = 0$ and

$$Z = \begin{pmatrix} \rho & 0 \\ 0 & O^{(m-1,n-1)} \end{pmatrix}, \quad 0 \leqslant \rho < 1.$$

Then, we have

$$u(Z) = \frac{1}{V(\mathfrak{C}_{\rm I})} \int_{\mathfrak{C}_{\rm I}} \varphi(U) \frac{(1-\rho^2)^n}{|1-\rho \bar{u}_{11}|^{2n}} \dot{U}, \tag{2.4.3}$$

where

$$U = \begin{pmatrix} u \\ U_1 \end{pmatrix}, \quad u = (u_{11}, \cdots, u_{1n}). \tag{2.4.4}$$

(2.4.3) can be written as

$$u(Z) = \frac{1}{V(\mathfrak{C}_{\rm I})} \int_{u\bar{u}'=1} \tau(u) \frac{(1-\rho^2)^n}{|1-\rho \bar{u}_{11}|^{2n}} \dot{u} \tag{2.4.5}$$

with

$$\tau(u) = \int_{U_1} \varphi(U) \dot{U}_1, \tag{2.4.6}$$

where U_1 runs over $(m-1) \times n$ matrices satisfying

$$\begin{pmatrix} u \\ U_1 \end{pmatrix} \overline{\begin{pmatrix} u \\ U_1 \end{pmatrix}}' = I^{(m)}. \tag{2.4.7}$$

Here u is considered as parameters satisfying $u\bar{u}' = 1$.

For any fixed u, the set of U_1 satisfying (2.4.7) forms a manifold of which the volume element is denoted by $\dot{U}_1$. It is easy to see that $\tau(u)$ is a continuous function on $u\bar{u}' = 1$. We apply Theorem 1.3.2 to (2.4.5), then we have

$$\lim_{\rho \to 1} u(Z) = \frac{\omega_{2n-1}}{V(\mathfrak{C}_{\rm I})} \tau(1, 0, \cdots, 0). \tag{2.4.8}$$

Notice that, by (2.2.4),

$$\frac{\omega_{2n-1}}{V(\mathfrak{C}_{\rm I}(m,n))} = \frac{1}{V(\mathfrak{C}_{\rm I}(m-1,n-1))}.$$

Substituting $u = (1, 0, \cdots, 0)$ into (2.4.7) we have

$$U_1 = (0, U_0), \quad U_0 = U_0^{(m-1,n-1)}$$

and $U_0 \bar{U}'_0 = I^{(m-1)}$. Consequently, we have, from (2.4.6) and (2.4.8),

$$\lim_{\rho \to 1} u(Z) = \frac{1}{V(\mathfrak{C}_{\rm I}(m-1,n-1))} \int_{\mathfrak{C}_{\rm I}(m-1,n-1)} \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & U_0 \end{pmatrix}\right) \dot{U}_0. \tag{2.4.9}$$

If $Z_0 \neq 0$, then by Theorem 2.3.2 there is a transformation $W = \Phi(Z)$ of Γ^{I} carrying

$$\begin{pmatrix} 1 & 0 \\ 0 & Z_0 \end{pmatrix} \text{into} \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix} \text{and} \begin{pmatrix} \rho & 0 \\ 0 & Z_1 \end{pmatrix} \text{into} \begin{pmatrix} \rho & 0 \\ 0 & W_1 \end{pmatrix},$$

where

$$W_1 = A_1(Z_1 - Z_0)(I - \bar{Z}'_0 Z)^{-1} D_1^{-1} \tag{2.4.10}$$

with

$$\bar{A}'_1 A_1 = (I - Z_0 \bar{Z}'_0)^{-1}, \quad \bar{D}'_1 D_1 = (I - \bar{Z}'_0 Z_0)^{-1}.$$

For $U \in \mathfrak{C}_{\mathrm{I}}$, we have

$$V = \Phi(U) \in \mathfrak{C}_{\mathrm{I}}$$

and

$$Z = \Phi^{-1}(W).$$

It is known that

$$\dot{V} = |\det(B\bar{U}' + A)|^{-2\pi} \dot{U}.$$

According to Theorem 2.2.2, we have, for $Z = \begin{pmatrix} \rho & 0 \\ 0 & Z_0 \end{pmatrix}$,

$$\begin{aligned} &\lim_{\rho \to 1} \int_{\mathfrak{C}_{\mathrm{I}}(m,n)} \varphi(U) P_{\mathrm{I}}(Z, U) \dot{U} \\ &= \lim_{\rho \to 1} \frac{1}{V(\mathfrak{C}_{\mathrm{I}}(m,n))} \int_{\mathfrak{C}_{\mathrm{I}}(m,n)} \varphi(\Phi^{-1}(V)) \frac{(1-\rho^2)^n}{|1 - \rho \bar{v}_{11}|^{2n}} \dot{V} \\ &= \frac{1}{V(\mathfrak{C}_{\mathrm{I}}(m-1, n-1))} \int_{\mathfrak{C}_{\mathrm{I}}(m-1,n-1)} \varphi\left(\Phi^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & V_0 \end{pmatrix}\right)\right) \dot{V}_0. \end{aligned}$$

Notice that the above formula holds uniformly with respect to Z_0 since in the proof of (2.4.9), for any given $\varepsilon > 0$ there is a ρ depending only on the upper bound of $|\varphi|$ such that (cf. Th. 1.3.2)

$$\begin{aligned} &\left| \frac{1}{V(\mathfrak{C}_{\mathrm{I}}(m,n))} \int_{\mathfrak{C}_{\mathrm{I}}(m,n)} \varphi(\Phi^{-1}(V)) \frac{(1-\rho^2)^n}{|1 - \rho \bar{v}_{11}|^{2n}} \dot{V} \right. \\ &\left. - \frac{1}{V(\mathfrak{C}_{\mathrm{I}}(m-1, n-1))} \int_{\mathfrak{C}_{\mathrm{I}}(m-1,n-1)} \varphi\left(\Phi^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & V_0 \end{pmatrix}\right)\right) \dot{V}_0 \right| < \varepsilon, \end{aligned}$$

where ρ is independent of Z_0.

By (2.4.10), we have

$$\Phi^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & V_0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & U_0 \end{pmatrix},$$

where

$$V_0 = A_1(U_0 - Z_0)(I - \bar{Z}'_0 U_0)^{-1} D_1^{-1}.$$

Hence

$$\dot{V}_0 = |\det(I - \bar{Z}'_0 U_0) A_1|^{-2(n-1)} \dot{U}_0 = \frac{\det(I - Z_0 \bar{Z}'_0)^{n-1}}{|\det(I - Z_0 \bar{U}'_0)|^{2(n-1)}} \dot{U}_0.$$

Finally we have

$$\begin{aligned} &\lim_{\rho \to 1} \int_{\mathfrak{C}_\mathrm{I}(m,n)} \varphi(U) P_\mathrm{I}(Z, U) \dot{U} \\ =& \frac{1}{V(\mathfrak{C}_\mathrm{I}(m-1, n-1))} \int_{\mathfrak{C}_\mathrm{I}(m-1,n-1)} \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & U_0 \end{pmatrix}\right) P_\mathrm{I}^{(m-1,n-1)}(Z_0, U_0) \dot{U}_0 \end{aligned}$$

uniformly with respect to Z_0.

Theorem 2.4.2 *Let $\varphi(U)$ be a real-valued function continuous on the characteristic manifold $\mathfrak{C}_\mathrm{I}(m,n)$ of $\mathfrak{R}_\mathrm{I}(m,n)$.*

(i) *If $Q \in \mathfrak{C}_\mathrm{I}^{(r)} (0 < r < m)$, the limit*

$$\lim_{Z \to Q} \int_{\mathfrak{C}_\mathrm{I}} \varphi(U) P_\mathrm{I}(Z, U) \dot{U}$$

exists and defines a function continuous in $\mathfrak{C}_\mathrm{I}^{(r)} \cong \mathfrak{R}_\mathrm{I}(r, n-m+r) \times \mathfrak{M}_\mathrm{I}^{(m-r)}$; besides, it is harmonic with respect to the coordinate of $\mathfrak{R}_\mathrm{I}(r, n-m+r)$. More precisely, if we set

$$Q = U_0' \begin{pmatrix} I^{(m-r)} & 0 \\ 0 & Z_1 \end{pmatrix} V_0,$$

the function

$$u(Z_1, U_0, V_0) = \lim_{Z \to Q} \int_{\mathfrak{C}_\mathrm{I}} \varphi(U) P_\mathrm{I}(Z, U) \dot{U}$$

is harmonic in $\mathfrak{R}_\mathrm{I}(r, n-m+r)$ with respect to the variable Z_1 for any $(U_0, V_0) \in \mathfrak{M}_\mathrm{I}^{(m-r)}$.

(ii) *For $Q \in \mathfrak{C}_\mathrm{I} = \mathfrak{C}_\mathrm{I}(m,n)$, then*

$$\lim_{Z \to Q} \int_{\mathfrak{C}_\mathrm{I}} \varphi(U) P_\mathrm{I}(Z, U) \dot{U} = \varphi(Q).$$

Proof (i) *When* $0 < r < m$, *we let*

$$Z = U_0' \begin{pmatrix} \Lambda^{(m-r)} & 0 \\ 0 & Z_1 \end{pmatrix} V_0,$$

where

$$\Lambda^{(m-r)} = [\lambda_1, \cdots, \lambda_{m-r}], \quad 1 > \lambda_1 \geqslant \cdots \geqslant \lambda_{m-r} \geqslant 0.$$

Applying the previous theorem repeatedly, we see that

$$\begin{aligned} & \lim_{\lambda_{m-r}\to 1} \cdots \lim_{\lambda_1 \to 1} \int_{\mathfrak{C}_\mathrm{I}(m,n)} \varphi(U) P_\mathrm{I}(Z,U)\dot{U} \\ = & \lim_{\lambda_{m-r}\to 1} \cdots \lim_{\lambda_1 \to 1} \int_{\mathfrak{C}_\mathrm{I}(m,n)} \varphi(U_0' U V_0) P_\mathrm{I} \left(\begin{pmatrix} \Lambda^{(m-r)} & 0 \\ 0 & Z_1 \end{pmatrix}, U \right) \dot{U} \\ = & \frac{1}{V(\mathfrak{C}_\mathrm{I}(r, n-m+r))} \int_{\mathfrak{C}_\mathrm{I}(r,n-m+r)} \varphi\left(U_0' \begin{pmatrix} I^{(m-1)} & 0 \\ 0 & U_1 \end{pmatrix} V_0 \right) \\ & \times \frac{\det(I - Z_1 \bar{Z}'_1)^{n-m+r}}{|\det(I - Z_1 \bar{U}'_1)|^{2(n-m+r)}} \dot{U}_1 \end{aligned} \tag{2.4.11}$$

uniformly with respect to Z_1, U_0, V_0.

Obviously the above expression is a continuous function of Z_1, U_0, V_0 and for any U_0 and V_0 it is harmonic in $\mathfrak{R}_\mathrm{I}(r, n-m+r)$.

Now we take in $\mathfrak{R}_\mathrm{I}$ an arbitrary sequence of points $Z_1, Z_2, \cdots, Z_k, \cdots$, which approaches Q.

We write

$$u(Z) = \int_{\mathfrak{C}_\mathrm{I}} \varphi(U) P_\mathrm{I}(Z,U)\dot{U},$$

and $u_r(Q)$ as the limiting function of (2.4.11).

For any given $\varepsilon > 0$, since $u_r(Q)$ is continuous, there is a neighbourhood $\mathfrak{B}(Q)$ of Q in the space of the mn complex variables $Z = (z_{j\alpha})$ such that for any point $P \in \mathfrak{B}(Q) \bigcap \mathfrak{C}_\mathrm{I}^{(r)}$, we always have

$$|u_r(P) - u_r(Q)| < \frac{\varepsilon}{2}.$$

Each point of Z_k has a representation

$$Z_k = U_k' \begin{pmatrix} \Lambda_k & 0 \\ 0 & Z_{0k} \end{pmatrix} V_k,$$

where

$$\Lambda_k = [\lambda_1^{(k)}, \cdots, \lambda_{m-r}^{(k)}], \quad 1 > \lambda_1^{(k)} \geqslant \cdots \geqslant \lambda_{m-r}^{(k)} \geqslant 0.$$

Since $Z_k \to Q$, we must have $\Lambda_k \to I^{(m-r)}$. This implies that when we take k sufficiently large, $\lambda_1^{(k)}, \cdots, \lambda_{m-r}^{(k)}$ can be as near to 1 as we please. Since (2.4.11) holds uniformly for all $Q \in \mathfrak{C}_{\mathrm{I}}^{(r)}$, we see that for $Q_k = U_k' \begin{pmatrix} I & 0 \\ 0 & Z_{0k} \end{pmatrix} V_k \in \mathfrak{C}_{\mathrm{I}}^{(r)}$,

$$|u(Z_k) - u_r(Q_k)| < \frac{\varepsilon}{2},$$

when k is sufficiently large.

Obviously $Q_k \to Q$, hence we can take k so large that $Q_k \in \mathfrak{B}(Q) \bigcap \mathfrak{C}_{\mathrm{I}}^{(r)}$. Then

$$|u(Z_k) - u_r(Q)| \leqslant |u(Z_k) - u_r(Q_k)| + |u_r(Q_k) - U_r(Q)| < \varepsilon.$$

This shows that for any sequence of points $Z_k \to Q$, the part (i) of our theorem holds.

(ii) When $Q \in \mathfrak{C}_{\mathrm{I}}$, we set

$$Q = U_0'(I^{(m)}, 0)V_0.$$

and

$$Z = U_0'(\Lambda, 0)V_0, \quad \Lambda = [\lambda_1, \cdots, \lambda_m].$$

Then, by Theorem 2.4.1 and Theorem 1.3.2,

$$\begin{aligned}
&\lim_{\lambda_m \to 1} \cdots \lim_{\lambda_1 \to 1} \int_{\mathfrak{C}_{\mathrm{I}}(m,n)} \varphi(U) P_{\mathrm{I}}(Z, U) \dot{U} \\
=& \lim_{\lambda_m \to 1} \frac{1}{\omega_{2(n-m)+1}} \int_{u\bar{u}'=1} \varphi\left(U_0' \begin{pmatrix} I^{(m-1)} & 0 \\ 0 & u \end{pmatrix} V_0 \right) \frac{(1-\lambda_m^2)^{n-m+1}}{|1-\lambda_m \bar{u}_1|^{2(n-m+1)}} \dot{u} \\
=& \varphi(U_0'(I^{(m)}, 0)V_0)) = \varphi(Q).
\end{aligned}$$

where $u = (u_1, \cdots, u_{n-m+1})$ with $u\bar{u}' = 1$.

As in the proof of part (i), for any sequence of points $Z_1, Z_2, \cdots Z_k \to Q$ in $\mathfrak{R}_{\mathrm{I}}$, we always have

$$\lim_{Z_k \to Q} \int_{\mathfrak{C}_{\mathrm{I}}} \varphi(U) P_{\mathrm{I}}(Z, U) \dot{U} = \varphi(Q).$$

The theorem is proved.

§2.5. A Dirichlet problem of $\mathfrak{R}_{\mathrm{I}}$

A real-valued function $u(Z)$ is said to be harmonic on the closure of $\mathfrak{R}_{\mathrm{I}}$, if it is continuous in $\mathfrak{B}_{\mathrm{I}}^{(0)}$, and on each $\mathfrak{C}_{\mathrm{I}}^{(r)} = \mathfrak{B}_{\mathrm{I}}^{(m-r)} - \mathfrak{B}_{\mathrm{I}}^{(m-r+1)} (r = 1, \cdots, m)$ it is

harmonic with respect to the coordinate of the base space $\mathfrak{R}_{\mathrm{I}}(r, n-m+r)$ of the slit space $\mathfrak{B}_{\mathrm{I}}^{(m-r)}$.

Notice that the differential equation on $\mathfrak{C}_{\mathrm{I}}^{(r)}$ can also be considered as a consequence of the differential equation

$$\operatorname{tr}((I - Z\overline{Z'})\bar{\partial}_Z(I - \overline{Z'}Z)\partial'_Z) \circ u = 0. \tag{2.5.1}$$

In fact, for $Z = \begin{pmatrix} I & 0 \\ 0 & Z_1 \end{pmatrix}$, the previous equation resuces to

$$\operatorname{tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & I - Z_1\overline{Z'}_1 \end{pmatrix}\begin{pmatrix} * & * \\ * & \bar{\partial}_{Z_1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I - \overline{Z'}_1 Z_1 \end{pmatrix}\begin{pmatrix} * & * \\ * & \partial'_{Z_1} \end{pmatrix}\right) \circ u = 0,$$

i.e.,

$$\begin{aligned}
&\operatorname{tr}\left(\begin{pmatrix} 0 & 0 \\ * & (I - Z_1\bar{Z'}_1)\bar{\partial}_{Z_1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ * & (I - \overline{Z'}_1 Z_1)\partial'_{Z_1} \end{pmatrix}\right) \circ u \\
=&\operatorname{tr}((I - Z_1\overline{Z'}_1)\bar{\partial}_{Z_1}(I - \overline{Z'}_1 Z_1)\partial'_{Z_1}) \circ u = 0.
\end{aligned}$$

Similarly for the case $Z = U'\begin{pmatrix} I & 0 \\ 0 & Z_1 \end{pmatrix}V$ with unitary U and V.

Now we formulate the following Dirichlet problem:

Given a continuous function $\varphi(U)$ on $\mathfrak{C}_{\mathrm{I}}$, whether there exists a unique function harmonic on the closure of $\mathfrak{R}_{\mathrm{I}}$, which takes the given boundary value $\varphi(U)$ on $\mathfrak{C}_{\mathrm{I}}$.

The answer is positive. We have

Theorem 2.5.1 *Let $\varphi(U)$ be a real-valued continuous function on $\mathfrak{C}_{\mathrm{I}}$. Then*

$$u(Z) = \int_{\mathfrak{C}_{\mathrm{I}}} \varphi(U) P_{\mathrm{I}}(Z, U)\dot{U} \tag{2.5.2}$$

is the unique function harmonic on the closure of $\mathfrak{R}_{\mathrm{I}}$, which takes the given boundary value $\varphi(U)$ on $\mathfrak{C}_{\mathrm{I}}$.

Proof The existence of the problem follows from Theorem 2.4.2; we remain to prove the uniqueness of the theorem.

Suppose that there is another function $u_1(Z)$ which is harmonic on the closure of $\mathfrak{R}_{\mathrm{I}}$ and which takes the given boundary value $\varphi(U)$ on $\mathfrak{C}_{\mathrm{I}}$. Then $u_1(Z) - u(Z)$ is again a function which is harmonic on the closure of $\mathfrak{R}_{\mathrm{I}}$ and vanishes on $\mathfrak{C}_{\mathrm{I}}$. The principle of extremity, Theorem 1.2.2, asserts that it vanishes identically, Therefore we have the theorem.

Notice that the Poisson kernel, by Theorem 2.2.3, satisfies a system of m^2 partial differential equations

$$\bar{\partial}_Z(I-\bar{Z}'Z)\partial_Z' \circ P_{\mathrm{I}}(Z,U)=0.$$

Consequently, the function represented by the Poisson integral satisfies also a system of m^2 partial differential equations.

§2.6. Harmonic functions in $\mathfrak{R}_{\mathrm{II}}$

Let Z denote an n-rowed symmetric matrix of the form

$$Z=\begin{pmatrix} \sqrt{2}z_{11} & z_{12} & \cdots & z_{1n} \\ z_{12} & \sqrt{2}z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & & \vdots \\ z_{1n} & z_{2n} & \cdots & \sqrt{2}z_{nn} \end{pmatrix}, \tag{2.6.1}$$

and let $\mathfrak{R}_{\mathrm{II}}$ denote the domain

$$I-Z\bar{Z}>0. \tag{2.6.2}$$

It is a space of $\dfrac{1}{2}n(n+1)$ complex variables

$$z_{11},z_{12},\cdots,z_{1n},z_{22},\cdots,z_{2n},\cdots,z_{nn}.$$

We introduce the operator

$$\partial_Z=\begin{pmatrix} \sqrt{2}\dfrac{\partial}{\partial z_{11}} & \dfrac{\partial}{\partial z_{12}} & \cdots & \dfrac{\partial}{\partial z_{1n}} \\ \dfrac{\partial}{\partial z_{12}} & \sqrt{2}\dfrac{\partial}{\partial z_{22}} & \cdots & \dfrac{\partial}{\partial z_{2n}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial}{\partial z_{1n}} & \dfrac{\partial}{\partial z_{2n}} & \cdots & \sqrt{2}\dfrac{\partial}{\partial z_{nn}} \end{pmatrix}, \tag{2.6.3}$$

and

$$\Delta_{\mathrm{II}}=\mathrm{tr}((I-\bar{Z}Z)\partial_Z(I-Z\bar{Z})\bar{\partial}_Z) \tag{2.6.4}$$

or

$$\Delta_{\mathrm{II}}=\sum_{\alpha,\beta,\lambda,\mu=1}^{n}\left(\delta_{\lambda\mu}-\sum_{\sigma=1}^{n}P_{\lambda\sigma}P_{\mu\sigma}z_{\lambda\sigma}\bar{z}_{\mu\sigma}\right)$$
$$\times\left(\delta_{\alpha\beta}-\sum_{\gamma=1}^{n}P_{\alpha\gamma}P_{\beta\gamma}z_{\alpha\gamma}\bar{z}_{\beta\gamma}\right)P_{\lambda\alpha}P_{\mu\beta}\frac{\partial^2}{\partial_{z\lambda\alpha}\partial\bar{z}_{\mu\beta}},$$

where $z_{\alpha\beta} = z_{\beta\alpha}$ and

$$P_{\alpha\beta} = \begin{cases} \sqrt{2}, & \text{for } \alpha = \beta, \\ 1, & \text{for } \alpha \neq \beta. \end{cases} \tag{2.6.5}$$

A real-valued function $u(Z)$ possessing continuous second derivatives is said to be harmonic in $\mathfrak{R}_{\rm II}$, if it satisfies in $\mathfrak{R}_{\rm II}$ the differential equation

$$\Delta_{\rm II} \circ u(Z) = 0. \tag{2.6.6}$$

Theorem 2.6.1 *If $u(Z)$ is harmonic in $\mathfrak{R}_{\rm II}$, it remains to be harmonic in $\mathfrak{R}_{\rm II}$, after a transformation of the group $\Gamma^{\rm II}$ of the motions of $\mathfrak{R}_{\rm II}$.*

Proof It is known that a transformation of $\mathfrak{R}_{\rm II}$ is of the form

$$W = (AZ + B)(\bar{B}Z + \bar{A})^{-1}, \tag{2.6.7}$$

where

$$A'\bar{B} = \bar{B}'A, \quad \bar{A}'A - B'\bar{B} = I.$$

Now we have

$$dW = (\bar{B}Z + \bar{A})'^{-1} dZ (\bar{B}Z + \bar{A})^{-1}.$$

If we use $a_{\alpha\beta} (1 \leqslant \alpha, \beta \leqslant n)$ to denote the elements of the matrix $(\bar{B}Z + \bar{A})^{-1}$, then the above relation can be written as

$$P_{\alpha\beta} dw_{\alpha\beta} = \sum_{\lambda,\mu=1}^{n} a_{\lambda\alpha} P_{\lambda\mu} dz_{\lambda\mu} a_{\mu\beta}.$$

Since $z_{\lambda\mu} = z_{\mu\lambda}$, we have

$$\frac{\partial w_{\alpha\beta}}{\partial z_{\lambda\mu}} = \begin{cases} \dfrac{1}{P_{\alpha\beta}} (a_{\lambda\alpha} a_{\mu\beta} + a_{\mu\alpha} a_{\lambda\beta}), & (\lambda \neq \mu) \\ \dfrac{1}{P_{\alpha\beta}} \sqrt{2} a_{\lambda\alpha} a_{\mu\beta}, & (\lambda = \mu). \end{cases}$$

For $\lambda \neq \mu$, we have

$$\begin{aligned} \frac{\partial}{\partial z_{\lambda\mu}} &= \sum_{\alpha \leqslant \beta} \frac{\partial w_{\alpha\beta}}{\partial z_{\lambda\mu}} \frac{\partial}{\partial w_{\alpha\beta}} \\ &= \sum_{\alpha<\beta} (a_{\lambda\alpha} a_{\mu\beta} + a_{\mu\alpha} a_{\lambda\beta}) \frac{\partial}{\partial w_{\alpha\beta}} + \sum_{\alpha=1}^{n} \frac{1}{2} (a_{\lambda\alpha} a_{\mu\alpha} + a_{\mu\alpha} a_{\lambda\alpha}) \sqrt{2} \frac{\partial}{\partial w_{\alpha\alpha}} \\ &= \sum_{\alpha<\beta} a_{\lambda\alpha} a_{\mu\beta} P_{\alpha\beta} \frac{\partial}{\partial w_{\alpha\beta}} + \sum_{\alpha>\beta} a_{\lambda\alpha} a_{\mu\beta} P_{\alpha\beta} \frac{\partial}{\partial w_{\alpha\beta}} + \sum_{\alpha=\beta} a_{\lambda\alpha} a_{\lambda\beta} P_{\alpha\beta} \frac{\partial}{\partial w_{\alpha\beta}} \end{aligned}$$

$$= \sum_{\alpha,\beta=1}^{n} a_{\lambda\alpha}\left(P_{\alpha\beta}\frac{\partial}{\partial w_{\alpha\beta}}\right)a_{\mu\beta}.$$

For $\lambda=\mu$, we have

$$\begin{aligned}\sqrt{2}\frac{\partial}{\partial z_{\lambda\lambda}} =&\sqrt{2}\sum_{\alpha\leqslant\beta}\frac{\partial w_{\alpha\beta}}{\partial z_{\lambda\lambda}}\frac{\partial}{\partial w_{\alpha\beta}}\\ =&\sum_{\alpha<\beta}2a_{\lambda\alpha}a_{\lambda\beta}\frac{\partial}{\partial w_{\alpha\beta}}+\sum_{\alpha=1}^{n}\sqrt{2}a_{\lambda\alpha}a_{\lambda\alpha}\frac{\partial}{\partial w_{\alpha\alpha}}\\ =&\sum_{\alpha<\beta}a_{\lambda\alpha}a_{\lambda\beta}\frac{\partial}{\partial w_{\alpha\beta}}+\sum_{\alpha>\beta}a_{\lambda\alpha}a_{\lambda\beta}P_{\alpha\beta}\frac{\partial}{\partial w_{\alpha\beta}}+\sum_{\alpha=\beta}a_{\lambda\alpha}a_{\lambda\beta}P_{\alpha\beta}\frac{\partial}{\partial w_{\alpha\beta}}\\ =&\sum_{\alpha\cdot\beta=1}^{n}a_{\lambda\alpha}\left(P_{\alpha\beta}\frac{\partial}{\partial w_{\alpha\beta}}\right)a_{\lambda\beta}.\end{aligned}$$

Consequently, we can put

$$P_{\lambda\mu}\frac{\partial}{\partial z_{\lambda\mu}} = \sum_{\alpha,\beta=1}^{n} a_{\lambda\alpha}\left(P_{\alpha\beta}\frac{\partial}{\partial w_{\alpha\beta}}\right)a_{\mu\beta}$$

for $\lambda,\mu=1,\cdots,n$, or

$$\partial_Z = (\bar{B}Z+\bar{A})^{-1}\partial_W(\bar{B}Z+\bar{A})'^{-1}.$$

On the other hand

$$\begin{aligned}I-W\bar{W} =&I-(\overline{B}Z+\overline{A})'^{-1}(AZ+B)'(\overline{A}\,\overline{Z}+\overline{B})(B\overline{Z}+A)^{-1}\\ =&(\overline{B}Z'+A)'^{-1}(I-Z\overline{Z})(B\overline{Z}+A)^{-1}.\end{aligned} \tag{2.6.8}$$

We have finally

$$\begin{aligned}&(I-\overline{Z}Z)\partial_Z(I-Z\overline{Z})\bar{\partial}_Z\\ =&(\overline{Z}B'+A')[(I-\overline{W}W)\partial_W(I-W\overline{W})\overline{\partial}_W](\overline{Z}B'+A')^{-1}.\end{aligned}$$

This leads to

$$\operatorname{tr}((I-\overline{W}W)\partial_W(I-W\overline{W})\overline{\partial}_W) = \operatorname{tr}((I-\overline{Z}Z)\partial_Z(I-Z\overline{Z})\overline{\partial}_Z),$$

which proves our theorem.

The Poisson kernel of $\mathfrak{R}_{\rm II}$ is known (Hua [4]) to be

$$P_{\rm II}(Z,S) = \frac{1}{V(\mathfrak{C}_{\rm II})}\cdot\frac{\det(I-Z\overline{Z})^{\frac{1}{2}(n+1)}}{|\det(I-Z\overline{Z})|^{n+1}}, \tag{2.6.9}$$

where S is an n-rowed symmetric unitary matrix, i.e., $S\bar{S} = I$ and is expressed as

$$S = \begin{pmatrix} \sqrt{2}s_{11} & s_{12} & \cdots & s_{1n} \\ s_{12} & \sqrt{2}s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & & \vdots \\ s_{1n} & s_{2n} & \cdots & \sqrt{2}s_{nn} \end{pmatrix} \tag{2.6.10}$$

Further, we have

$$V(\mathfrak{C}_{\mathrm{II}}) = 2^{\frac{1}{2}n(n+1)}\pi^{\frac{1}{4}n(n+1)} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \prod_{\nu=1}^{n-1} \frac{\Gamma\left(\frac{1}{2}(n-\nu)+1\right)}{\Gamma(n-\nu+1)}.$$

Since $\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)$, we have①

$$\frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma(\mu+1)} = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\mu+1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma(\mu+1)} = \frac{\sqrt{\pi}}{2^{\mu}\Gamma\left(\frac{\mu+1}{2}\right)},$$

and then

$$\begin{aligned} V(\mathfrak{C}_{\mathrm{II}}) &= 2^{\frac{1}{2}n(n+1)}\pi^{\frac{1}{4}n(n+1)} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \prod_{\mu=1}^{n-1} \frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma(\mu+1)} \\ &= 2^{\frac{1}{2}n(n+1)}\pi^{\frac{1}{4}n(n+1)} \frac{\pi^{\frac{\pi}{2}}}{2^{\frac{1}{2}n(n-1)}} \prod_{\mu=1}^{n} \frac{1}{\Gamma\left(\frac{\mu+1}{2}\right)} \\ &= \prod_{\mu=1}^{n} \frac{2\pi^{\frac{1}{2}(\mu+1)}}{\Gamma\left(\frac{\mu+1}{2}\right)} = \prod_{\mu=1}^{n} \omega_{\mu}. \end{aligned} \tag{2.6.11}$$

Theorem 2.6.2 *After the transformation* (2.6.7), *the Poisson kernel becomes*

$$P_{\mathrm{II}}(Z,S) = P_{\mathrm{II}}(W,T)|\det(B\overline{S}+A)|^{-(n+1)},$$

where

$$T = (AS+B)(\overline{B}S+\overline{A})^{-1}. \tag{2.6.12}$$

① The constant differs from the original one by a factor 2^{n^2} (cf. the footnote on p. 1046).

Proof Clearly

$$I - W\overline{T} = (\overline{B}Z + \overline{A})'^{-1}(I - Z\overline{S})(B\overline{S} + A)^{-1}.$$

Combining with (2.6.8), we have immediately

$$\frac{\det(I - Z\overline{Z})^{\frac{1}{2}(n+1)}}{|\det(I - Z\overline{S})|^{n+1}} = \frac{\det(I - W\overline{W})^{\frac{1}{2}(n+1)}|\det(\overline{B}Z + \overline{A})|^{n+1}}{|\det((I - W\overline{T})(\overline{B}Z + \overline{A})(B\overline{S} + A))|^{n+1}}$$

$$= \frac{\det(I - W\overline{W})^{\frac{1}{2}(n+1)}}{|\det(I - W\overline{T})|^{n+1}}|\det(B\overline{S} + A)|^{-(n+1)}.$$

Theorem 2.6.3 *The Poisson kernel $P_{\mathrm{II}}(Z, S)$ is harmonic in $\mathfrak{R}_{\mathrm{II}}$ with respect to the variable Z.*

Proof According to Theorems 2.6.1 and 2.6.2 and the method used in the proof of Theorem 2.2.3, it is sufficient to prove that

$$[\Delta_{\mathrm{II}} \circ P_{\mathrm{II}}(Z, S)]_{Z=0} = 0.$$

By (2.6.4) and (2.6.9), we have

$$[\Delta_{\mathrm{II}} \circ P_{\mathrm{II}}(Z, S)]_{Z=0}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{II}})}\left[\sum_{\alpha.\lambda=1}^{n} P_{\lambda\alpha}P_{\lambda\alpha}\frac{\partial^2}{\partial z_{\lambda\alpha}\partial\bar{z}_{\lambda\alpha}}\det(I - Z\bar{Z})^{\frac{n+1}{2}}\det(I - Z\bar{S})^{-\frac{n+1}{2}}\right.$$

$$\left.\times \det(I - \bar{Z}S)^{-\frac{n+1}{2}}\right]_{Z=0}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{II}})}\left[2\sum_{\lambda\leqslant\alpha}\frac{\partial^2}{\partial z_{\lambda\alpha}\partial\bar{z}_{\lambda\alpha}}\left\{1 - (n+1)\sum_{\beta\leqslant\gamma}|z_{\beta\gamma}|^2 + \cdots\right\}\right.$$

$$\left.\times\left\{1 + (n+1)\sum_{\beta\leqslant\gamma} z_{\beta\gamma}\bar{S}_{\beta\gamma} + \cdots\right\}\left\{1 + (n+1)\sum_{\beta\leqslant\gamma}\bar{z}_{\beta\gamma}S_{\beta\gamma} + \cdots\right\}\right]_{Z=0}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{II}})}\left[-2(n+1)\sum_{\lambda\leqslant\alpha} 1 + 2(n+1)^2\sum_{\lambda\leqslant\alpha}\bar{S}_{\lambda\alpha}S_{\lambda\alpha}\right]$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{II}})}\left[-n(n+1)^2 + (n+1)^2\sum_{\lambda,\alpha=1}^{n}|P_{\lambda\alpha}S_{\lambda\alpha}|^2\right]$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{II}})}[-n(n+1)^2 + (n+1)^2 n] = 0.$$

Consequently, for any real-valued function $\varphi(S)$ continuous on $\mathfrak{C}_{\rm II}$, the Poisson integral

$$u(Z) = \int_{\mathfrak{C}_{\rm II}} \varphi(S) P_{\rm II}(Z, S) \dot{S}$$

defines a harmonic function in $\mathfrak{R}_{\rm II}$.

§2.7. The boundary properties of the Poisson integral of $\mathfrak{R}_{\rm II}$

More precisely, we use $\mathfrak{R}_{\rm II}(n)$ and $\mathfrak{C}_{\rm II}(n)$ to denote $\mathfrak{C}_{\rm II}$ and $\mathfrak{C}_{\rm II}$ respectively. Let $\mathfrak{B}_{\rm II}^{(n-r)}$ be the set of symmetric matrices Z such that $I - Z\bar{Z}$ is positive semi-definite and of rank $\leqslant r (0 \leqslant r \leqslant n)$. Evidently $\mathfrak{B}_{\rm II}^{(0)}$ is the closure of $\mathfrak{R}_{\rm II}$ and $\mathfrak{B}_{\rm II}^{(n)}$ is the characteristic manifold $\mathfrak{C}_{\rm II}$ of $\mathfrak{R}_{\rm II}$.

Analogous to Theorem 2.3.1, we have

Theorem 2.7.1 *We define $\mathfrak{C}_{\rm II}^{(r)} = \mathfrak{B}_{\rm II}^{(n-r)} - \mathfrak{B}_{\rm II}^{(n-r+1)}$. Then $\mathfrak{C}_{\rm II}^{(r)}$ is an invariant subspace of $\mathfrak{B}_{\rm II}^{(0)}$ and is transitive under the group $\Gamma^{\rm II}$.*

Further we have

Theorem 2.7.2 *The sequence of sets*

$$\mathfrak{B}_{\rm II}^{(0)} \supset \mathfrak{B}_{\rm II}^{(1)} \supset \cdots \supset \mathfrak{B}_{\rm II}^{(n-1)}$$

forms a chain of slit spaces. The slit of $\mathfrak{B}_{\rm II}^{(n-1)}$ is $\mathfrak{C}_{\rm II}$. More precisely, $\mathfrak{C}_{\rm II}^{(r)} = \mathfrak{B}_{\rm II}^{(n-r)} - \mathfrak{B}_{\rm II}^{(n-r+1)}$ is homeomorphic to the topological product

$$\mathfrak{R}_{\rm II}(r) \times \mathfrak{M}_{\rm II}^{(n-r)},$$

where the set $\mathfrak{M}_{\rm II}^{(s)}$ can be defined as the following. Two n-rowed unitary matrices U and V are said to be equivalent if there exist an s-rowed real orthogonal matrix $\Gamma^{(s)}$ and an unitary matrix $U^{(n-s)}$ such that

$$U = \begin{pmatrix} \Gamma^{(s)} & 0 \\ 0 & U^{(n-s)} \end{pmatrix} V.$$

By equivalence, we classify the unitary matrices into classes. The totality of classes defines the set $\mathfrak{M}_{\rm II}^{(s)}$.

Proof It is known that each element Z of $\mathfrak{C}_{\rm II}^{(r)}$ can be expressed as

$$Z = U' \begin{pmatrix} I^{(n-r)} & 0 \\ 0 & W^{(r)} \end{pmatrix} U, \tag{2.7.1}$$

where $I - W\bar{W} > 0$.

From

$$Z = U_0' \begin{pmatrix} I & 0 \\ 0 & W_0 \end{pmatrix} U_0,$$

we deduce that

$$U_1' \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix} U_1 = \begin{pmatrix} I & 0 \\ 0 & W_0 \end{pmatrix},$$

where $U_1 = UU_0^{-1}$.

Consequently, we have

$$U_1 = \begin{pmatrix} \Gamma^{(n-r)} & 0 \\ 0 & U^{(r)} \end{pmatrix}.$$

The theorem follows.

Theorem 2.7.3 *Let $\varphi(S)$ be a continuous real-valued function defined on the characteristic manifold $\mathfrak{C}_{\mathrm{II}}(n)$ of $\mathfrak{R}_{\mathrm{II}}(n)$.*

(i) *Let*

$$Q = U_0' \begin{pmatrix} I^{(n-r)} & 0 \\ 0 & Z_1 \end{pmatrix} U_0$$

be an arbitrary point of $\mathfrak{C}_{\mathrm{II}}^{(r)} (0 < r < n)$, then

$$\lim_{z \to Q} \int_{\mathfrak{C}_{\mathrm{II}}(n)} \varphi(S) P_{\mathrm{II}}(Z, S) \dot{S}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{II}}(r))} \int_{\mathfrak{C}_{\mathrm{II}}(r)} \varphi \left(U_0' \begin{pmatrix} I^{(n-r)} & 0 \\ 0 & S_1 \end{pmatrix} U_0 \right) \frac{\det(I^{(r)} - Z_1 \bar{Z}_1)^{\frac{r+1}{2}}}{|\det(I - Z_1 \bar{S}_1)|^{r+1}} \dot{S}_1.$$

The last integral represents a function continuous in $\mathfrak{C}_{\mathrm{II}}^{(r)} (\cong \mathfrak{R}_{\mathrm{II}}(r) \times \mathfrak{M}_{\mathrm{II}}^{(n-r)})$ and harmonic with respect to Z_1 in $\mathfrak{R}_{\mathrm{II}}(r)$.

(ii) *If $S_0 \in \mathfrak{C}_{\mathrm{II}}(n)$, then*

$$\lim_{z \to S_0} \int_{\mathfrak{C}_{\mathrm{II}}} \varphi(S) P_{\mathrm{II}}(Z, S) \dot{S} = \varphi(S_0).$$

Proof It is sufficient to prove the following special case for $r = n - 1$ and $Z = \begin{pmatrix} \rho & 0 \\ 0 & O \end{pmatrix}$, that is

$$\lim_{\rho \to 1} \frac{1}{V(\mathfrak{C}_{\mathrm{II}}(n))} \int_{\mathfrak{C}_{\mathrm{II}}(n)} \varphi(S) \frac{(1 - \rho^2)^{\frac{n+1}{2}}}{|1 - \rho \bar{s}_{11}|^{n+1}} \dot{S}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{II}}(n-1))} \int_{\mathfrak{C}_{\mathrm{II}}(n-1)} \varphi \left(\begin{pmatrix} 1 & 0 \\ 0 & S_1 \end{pmatrix} \right) \dot{S}_1, \tag{2.7.2}$$

since the remaining part of the proof can be carried out by the same method used in §2.4.

Let

$$S = \begin{pmatrix} s_{11} & s \\ s' & S_1 \end{pmatrix}. \tag{2.7.3}$$

where $S = (s_{12}, \cdots, s_{1n})$, Write

$$s_{11} = x_1 + ix_2,\ s_{12} = x_3 e^{i\theta_1}, \cdots,\ s_{1n} = x_{n+1} e^{i\theta_{n-1}}. \tag{2.7.4}$$

From $|s_{11}|^2 + s\bar{s}' = 1$, it follows that $xx' = 1$, where $x = (x_1, \cdots, x_{n+1})$ is a real vector. Now we are going to prove that for a given x satisfying $xx' = 1$, we can construct a unitary symmetric matrix S, that is ,there exists an $(n-1) \times (n-1)$ symmetric matrix S_1 such that

$$s_{11}\bar{s} + s\bar{S}_1 = 0, \quad s'\bar{s} + S_1\bar{S}_1 = I. \tag{2.7.5}$$

Without loss of generality we take $\theta_1 = \cdots = \theta_{n-1} = 0$. There is a real orthogonal matrix $\Gamma = \Gamma^{(n-1)}$ such that

$$s = (\lambda, 0, \cdots, 0)\Gamma, \quad \lambda = \sqrt{x_3^2 + \cdots + x_{n+1}^2}.$$

Equations of (2.7.5) become

$$s_{11}(\lambda, 0, \cdots, 0) + (\lambda, 0, \cdots, 0)\bar{T} = 0,$$
$$(\lambda, 0, \cdots, 0)'(\lambda, 0, \cdots, 0) + T\bar{T} = I,$$

where $T = \Gamma' S_1 \Gamma$. Then

$$T = [\bar{s}_{11}, 1, \cdots, 1]$$

is a solution.

Now we take$(x_1, \cdots, x_{n+1})$as parameters of (2.7.5). For a fixed$(x_1, x_2, \cdots, x_{n+1})$, the manifold of (2.7.5) is denoted byΣ.Then the Poisson integral can be written as

$$\frac{1}{V(\mathfrak{C}_{II})}\int_{\mathfrak{C}_{II}} \varphi(S)\frac{(1-\rho^2)^{\frac{n+1}{2}}}{|1-\rho\bar{s}_{11}|^{n+1}}\dot{S} = \int_{xx'=1}\psi(x)\frac{(1-\rho^2)^{\frac{n+1}{2}}}{|1-\rho(x_1-ix_2)|^{n+1}}\dot{x},$$

where

$$\psi(x) = \frac{1}{V(\mathfrak{C}_{II}(n))}\int_{\Sigma}\varphi(S)\dot{\Sigma}.$$

According to Theorem 1.3.2, we have

$$\lim_{\rho\to 1}\int_{xx'=1}\psi(x)\frac{(1-\rho^2)^{\frac{n+1}{2}}}{|1-\rho(x_1-ix_2)|^{n+1}}\dot{x} = \omega_n\psi(1, 0, \cdots, 0).$$

Therefore

$$\lim_{\rho\to 1}\frac{1}{V(\mathfrak{C}_{\rm II}(n))}\int_{\mathfrak{C}_{\rm II}(n)}\varphi(S)\frac{(1-\rho^2)^{\frac{n+1}{2}}}{|1-\rho\bar{s}_{11}|^{n+1}}\dot{S}$$
$$=\frac{1}{V(\mathfrak{C}_{\rm II}(n-1))}\int_{S_1\in\mathfrak{C}_{\rm II}(n-1)}\varphi\left(\begin{pmatrix}1 & 0\\ 0 & S_1\end{pmatrix}\right)\dot{S}_1.$$

This proves (2.7.2).

§2.8. A Dirichlet problem of $\mathfrak{R}_{\rm II}$

A real-valued function $u(Z)$ is said to be harmonic on the closure of $\mathfrak{R}_{\rm II}$ if it is continuous on the characteristic manifold $\mathfrak{C}_{\rm II}(n)$ and on each $\mathfrak{C}_{\rm II}^{(r)}(r=1,\cdots,n)$ it is harmonic with respect to the coordinate of the base space $\mathfrak{R}_{\rm II}(r)$.

Similar to the proof of Theorem 2.5.1, we solve a corresponding Dirichlet problem for $\mathfrak{R}_{\rm II}$.

Theorem 2.8.1 *Given a real-valued continuous function $\varphi(S)$ on the characteristic manifold $\mathfrak{C}_{\rm II}$, the Poisson integral*

$$u(Z)=\int_{\mathfrak{C}_{\rm II}}\varphi(S)P_{\rm II}(Z,S)\dot{S}$$

gives the unique function which is harmonic on the closure of $\mathfrak{R}_{\rm II}$ and takes the given boundary value $\varphi(S)$ on $\mathfrak{C}_{\rm II}$.

Remark It can be proved as in §2.2 that the Poisson kernel satisfies a set of equations

$$\partial_Z(I-Z\overline{Z})\bar{\partial}_Z\circ P_{\rm II}(Z,U)=0,$$

and so does the function defined by the Poisson integral.

§2.9. Harmonic functions in $\mathfrak{R}_{\rm III}$

Let Z be an n-rowed skew-symmetric matrix

$$Z=\begin{pmatrix}0 & z_{12} & \cdots & z_{1n}\\ -z_{12} & 0 & \cdots & z_{2n}\\ \vdots & \vdots & & \vdots\\ -z_{1n} & -z_{2n} & \cdots & 0\end{pmatrix}, \tag{2.9.1}$$

and let $\mathfrak{R}_{\rm III}$ denote the domain

$$I+Z\bar{Z}>0 \tag{2.9.2}$$

which is a domain of $\frac{1}{2}n(n-1)$ complex variable $z_{12},\cdots,z_{1n},z_{23},\cdots,z_{2n},\cdots,z_{n-1,n}$.

We introduce the operators

$$\partial Z = \begin{pmatrix} 0 & \frac{\partial}{\partial z_{12}} & \cdots & \frac{\partial}{\partial z_{1n}} \\ -\frac{\partial}{\partial z_{12}} & 0 & \cdots & \frac{\partial}{\partial z_{2n}} \\ \vdots & \vdots & & \vdots \\ -\frac{\partial}{\partial z_{1n}} & -\frac{\partial}{\partial z_{2n}} & \cdots & 0 \end{pmatrix} \tag{2.9.3}$$

and

$$\Delta_{\mathrm{III}} = \mathrm{tr}((I + \bar{Z}Z)\partial_Z(I + Z\bar{Z})\bar{\partial}_Z), \tag{2.9.4}$$

i.e.,

$$\Delta_{\mathrm{III}} = \sum_{\lambda,\mu,\alpha,\beta=1}^{n} \left(\delta_{\lambda\mu} - \sum_{\sigma=1}^{n} z_{\lambda\sigma}\bar{z}_{\mu\sigma}\right)\left(\delta_{\alpha\beta} - \sum_{\gamma=1}^{n} z_{\alpha\gamma}\bar{z}_{\beta\gamma}\right) q_{\lambda\alpha}q_{\mu\beta}\frac{\partial^2}{\partial z_{\lambda\alpha}\partial\bar{z}_{\mu\beta}},$$

where $z_{\alpha\beta} = -z_{\beta\alpha}$ and

$$q_{\alpha\beta} = \begin{cases} 0 & \text{for } \alpha = \beta, \\ 1 & \text{for } \alpha \neq \beta. \end{cases} \tag{2.9.5}$$

A real-valued function $u(Z)$ possessing continuous second derivatives is said to be harmonic in $\mathfrak{R}_{\mathrm{III}}$ if it satisfies, in $\mathfrak{R}_{\mathrm{III}}$, the differential equation

$$\Delta_{\mathrm{III}} \circ u(Z) = 0. \tag{2.9.6}$$

Theorem 2.9.1 *If u(Z) is harmonic in* $\mathfrak{R}_{\mathrm{III}}$, *it remains to be harmonic in* $\mathfrak{R}_{\mathrm{III}}$ *after any transformation of the group* Γ^{III} of motions of $\mathfrak{R}_{\mathrm{III}}$

Proof It is known that the transformation of Γ^{III} is of the form

$$W = (AZ + B)(-\bar{B}Z + \bar{A})^{-1}, \tag{2.9.7}$$

where

$$A'\bar{B} = -\bar{B}'A, \quad A'\bar{A} - \bar{B}'B = I.$$

Differentiating (2.9.7), we have

$$dW = (-\bar{B}Z + \bar{A})'^{-1}dZ(-\bar{B}Z + \bar{A})^{-1}.$$

Let the elements of $(-\bar{B}Z + \bar{A})^{-1}$ be $b_{\alpha\beta}(\alpha, \beta = 1, \cdots, n)$. The previous equality can be written as

$$q_{\alpha\beta}dw_{\alpha\beta} = \sum_{\lambda,\mu=1}^{n} b_{\lambda\alpha}q_{\lambda\mu}dz_{\lambda\mu}b_{\mu\beta},$$

where $z_{\lambda\mu} = -z_{\mu\lambda}$. Then, we have

$$q_{\lambda\mu}\frac{\partial w_{\alpha\beta}}{\partial z_{\lambda\mu}} = q_{\alpha\beta}(b_{\lambda\alpha}b_{\mu\beta} - b_{\mu\alpha}b_{\lambda\beta}).$$

Hence

$$\begin{aligned} q_{\lambda\mu}\frac{\partial}{\partial z_{\lambda\mu}} &= \sum_{\alpha<\beta} q_{\lambda\mu}\frac{\partial w_{\alpha\beta}}{\partial z_{\lambda\mu}}\frac{\partial}{\partial w_{\alpha\beta}} \\ &= \sum_{\alpha<\beta} q_{\alpha\beta}b_{\lambda\alpha}b_{\mu\beta}\frac{\partial}{\partial w_{\alpha\beta}} - \sum_{\alpha<\beta} q_{\alpha\beta}b_{\lambda\beta}b_{\mu\alpha}\frac{\partial}{\partial w_{\alpha\beta}} \\ &= \sum_{\alpha,\beta=1}^{n} b_{\lambda\alpha}\left(q_{\alpha\beta}\frac{\partial}{\partial w_{\alpha\beta}}\right)b_{\mu\beta}, \end{aligned}$$

that is,

$$\partial_Z = (-\overline{B}Z + \overline{A})^{-1}\partial_W(-\overline{B}Z + \overline{A})'^{-1}.$$

Applying the formula

$$I + W\overline{W} = (-\overline{B}Z + \overline{A})'^{-1}(I + Z\overline{Z})(-B\overline{Z} + A)^{-1},$$

we have

$$\mathrm{tr}((I + \bar{Z}Z)\partial_Z(I + Z\bar{Z})\bar{\partial}_Z) = \mathrm{tr}((I + \bar{W}W)\partial_W(I + W\bar{W})\bar{\partial}_W).$$

This proves the theorem.

The Poisson kernel of $\mathfrak{R}_{\mathrm{III}}$ is equal to

$$P_{\mathrm{III}}(Z, K) = \frac{1}{V(\mathfrak{C}_{\mathrm{III}})}\cdot\frac{\det(I + Z\bar{Z})^a}{|\det(I + Z\bar{K})|^{2a}} \tag{2.9.8}$$

(Hua[4], Hua and Look[1]), where

$$a = \begin{cases} \dfrac{n-1}{2}, & \text{for even } n, \\ \dfrac{n}{2}, & \text{for odd } n, \end{cases} \tag{2.9.9}$$

and①

$$V(\mathfrak{C}_{\mathrm{III}}(n)) = \begin{cases} \displaystyle\prod_{\nu=1}^{\frac{1}{2}n} \omega_{4\nu-3}, & \text{for even } n, \\ \displaystyle\frac{1}{2\pi}\prod_{\nu=1}^{\frac{1}{2}(n+1)} \omega_{4\nu-3} = \prod_{\nu=2}^{\frac{1}{2}(n+1)} \omega_{4\nu-3}, & \text{for odd } n. \end{cases} \tag{2.9.10}$$

① The constant differs from the original one by the fact given in the footnote on p. 1046.

Moreover, the matrix K in (2.9.8) is of the form

$$K = U'F^{(n)}U, \tag{2.9.11}$$

where U is an n-rowed unitary matrix and

$$F^{(n)} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for even } n, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+} 0 & \text{for odd } n. \end{cases} \tag{2.9.12}$$

Theorem 2.9.2 *After the transformation* (2.9.7) *of* Γ^{III}, *the Poisson kernel becomes*

$$P_{\mathrm{III}}(Z, K) = P_{\mathrm{III}}(W, J)|\det(B\bar{K} + \bar{A})|^{-a},$$

where

$$J = (AK + B)(-\bar{B}K + \bar{A})^{-1}. \tag{2.9.13}$$

The proof is similar to that of Theorem 2.6.2.

Theorem 2.9.3 *The Poisson kernel* $P_{\mathrm{III}}(Z, K)$ *is harmonic in* $\mathfrak{R}_{\mathrm{III}}$ *with respect to the variable* Z.

Proof It is sufficient to prove that

$$[\Delta_{\mathrm{III}} \circ P_{\mathrm{III}}(Z, K)]_{Z=0} = 0.$$

In fact, let $K = (k_{\alpha\beta})$ and then

$$[\Delta_{\mathrm{III}} \circ P_{\mathrm{III}}(Z, K)]_{Z=0} = \left[\sum_{\lambda,\alpha=1}^{n} q_{\lambda\alpha}^2 \frac{\partial^2 P_{\mathrm{III}}(Z, K)}{\partial z_{\lambda\alpha}\partial\bar{z}_{\lambda\alpha}}\right]_{Z=0}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{III}})}\left[\sum_{\lambda,\alpha=1}^{n} q_{\lambda\alpha}^2 \frac{\partial^2}{\partial z_{\lambda\alpha}\partial\bar{z}_{\lambda\alpha}} \det(I + Z\bar{Z})^a \det(I + Z\bar{K})^{-a} \det(I + \bar{Z}K)^{-a}\right]_{Z=0}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{III}})}\left[\sum_{\lambda,\alpha=1}^{n} q_{\lambda\alpha}^2 \frac{\partial^2}{\partial z_{\lambda\alpha}\partial\bar{z}_{\lambda\alpha}} \left\{1 - 2a\sum_{\beta<\nu}|z_{\beta\nu}|^2 + \cdots\right\}\left\{1 + 2a\sum_{\beta<\nu} z_{\beta\nu}\bar{k}_{\beta\nu} + \cdots\right\}\right.$$

$$\left.\times\left\{1 + 2a\sum_{\beta<\nu}\bar{z}_{\beta\nu}k_{\beta\nu} + \cdots\right\}\right]_{Z=0}$$

$$= \frac{1}{V(\mathfrak{C}_{\mathrm{III}})}\left[(-2an)\sum_{\lambda,\alpha=1}^{n} q_{\lambda\alpha}^2 + 4a^2\sum_{\lambda,\alpha=1}^{n} k_{\lambda\alpha}\bar{k}_{\lambda\alpha}\right]$$

$$=\frac{1}{V(\mathfrak{C}_{\mathrm{III}})}[-2an(n-1)+4a^2\mathrm{tr}(K\bar{K}')]. \tag{2.9.14}$$

From (2.9.11) and (2.9.12), we have

$$\mathrm{tr}(K\bar{K}')=\begin{cases} n, & \text{for even } n,\\ n-1, & \text{for odd } n.\end{cases}$$

Then, by (2.9.9), it follows that (2.9.14) equals zero.

For any real-valued function $\varphi(K)$ continuous on $\mathfrak{C}_{\mathrm{III}}$, the Poisson integral

$$u(Z)=\int_{\mathfrak{C}_{\mathrm{III}}}\varphi(K)P_{\mathrm{III}}(Z,K)\dot{K}$$

defines a harmonic function in $\mathfrak{R}_{\mathrm{III}}$.

§2.10. The boundary properties of the Poisson integral of $\mathfrak{R}_{\mathrm{III}}$

More precisely, we use $\mathfrak{R}_{\mathrm{III}}(n)$ and $\mathfrak{C}_{\mathrm{III}}(n)$ to denote $\mathfrak{R}_{\mathrm{III}}$ and $\mathfrak{C}_{\mathrm{III}}$ respectively. Let $\mathfrak{B}_{\mathrm{III}}^{(2r)}$ be the set of all skew-symmetric matrices Z such that $I+Z\bar{Z}$ is positive semi-definite and of rank $\leqslant n-2r$ $\left(r=1,2,\cdots,\left[\frac{n}{2}\right]\right)$. Note that the rank of $I+Z\bar{Z}$ has the same parity as n. In fact, there is a unitary matrix U (Hua [1]) such that

$$UZU'=\begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix}\dot{+}\begin{pmatrix}0&\lambda_2\\-\lambda_2&0\end{pmatrix}\dot{+}\cdots\dot{+}\begin{pmatrix}0&\lambda_r\\-\lambda_r&0\end{pmatrix}\dot{+}O^{(n-2r)}.$$

Then $U(I+Z\bar{Z})\bar{U}'=[1-|\lambda_1|^2,1-|\lambda_1|^2,\cdots,1-|\lambda_r|^2,1-|\lambda_r|^2,1,\cdots,1]$. The assertion is therefore true.

Further $\mathfrak{B}_{\mathrm{III}}^{(0)}$ is the closure of $\mathfrak{R}_{\mathrm{III}}$ and $\mathfrak{B}_{\mathrm{III}}^{2[\frac{n}{2}]}$ is equal to $\mathfrak{C}_{\mathrm{III}}$.

We have analogously the following

Theorem 2.10.1 *We define* $\mathfrak{C}_{\mathrm{III}}^{(n-2r)}=\mathfrak{B}_{\mathrm{III}}^{(2r)}-\mathfrak{B}_{\mathrm{III}}^{(2r+2)}$. *Then* $\mathfrak{C}_{\mathrm{III}}^{(n-2r)}$ *is an invariant subspace of* $\mathfrak{B}_{\mathrm{III}}^{(0)}$ *and is transitive under the transformations of the group* Γ^{III}.

Theorem 2.10.2 *We have a chain of slit spaces*

$$\mathfrak{B}_{\mathrm{III}}^{(0)}\supset\mathfrak{B}_{\mathrm{III}}^{(2)}\supset\cdots\supset\mathfrak{B}_{\mathrm{III}}^{2[\frac{n}{2}]-2}.$$

The slit of $\mathfrak{B}_{\mathrm{III}}^{2[\frac{n}{2}]-2}$ *is* $\mathfrak{C}_{\mathrm{III}}$. *More precisely,* $\mathfrak{B}_{\mathrm{III}}^{(n-2r)}$ *is homeomorphic*

$$\mathfrak{R}_{\mathrm{III}}(n-2r)\times\mathfrak{M}_{\mathrm{III}}^{(2r)},$$

where $\mathfrak{M}_{\mathrm{III}}^{(2r)}$ *is a set defined as follows:*

Two n-rowed unitary matrices U and V are said to be equivalent if there exist a 2r-rowed unitary symplectic matrix $P^{(2r)}$ and a unitary matrix $U^{(n-2r)}$ such that

$$U = \begin{pmatrix} P^{(2r)} & 0 \\ 0 & U^{(n-2r)} \end{pmatrix} V.$$

By equivalence, we classify the unitary matrices into classes. The totality of classes defines the set $\mathfrak{M}_{\rm III}^{(2r)}$.

Proof It is known (Hua[1]) that each element Z of $\mathfrak{B}_{\rm III}^{(n-2r)}$ can be expressed as

$$Z = U' \begin{pmatrix} F^{(2r)} & 0 \\ 0 & W^{(n-2r)} \end{pmatrix} U,$$

where $I + W\bar{W} > 0$.

From

$$Z = U_0' \begin{pmatrix} F^{(2r)} & 0 \\ 0 & W_0 \end{pmatrix} U_0$$

we deduce that

$$U_1' \begin{pmatrix} F^{(2r)} & 0 \\ 0 & W \end{pmatrix} U_1 = \begin{pmatrix} F^{(2r)} & 0 \\ 0 & W_0 \end{pmatrix},$$

where $U_1 = UU_0^{-1}$.

Consequently, we have

$$U_1 = \begin{pmatrix} P^{(2r)} & 0 \\ 0 & U^{(n-2r)} \end{pmatrix}, \quad P'FP = F^{(2r)}.$$

The theorem follows.

Theorem 2.10.3 *Let $\varphi(K)$ be a continuous real-valued function defined on the characteristic manifold $\mathfrak{C}_{\rm III}(n)$ of $\mathfrak{R}_{\rm III}(n)$*

(i) *Let*

$$Q = U_0' \begin{pmatrix} F^{(2r)} & 0 \\ 0 & Z_0 \end{pmatrix} U_0$$

be an arbitrary point on $\mathfrak{B}_{\rm III}^{(n-2r)}$ $\left(0 < r < \left[\frac{n}{2}\right]\right)$, then

$$\lim_{Z\to Q} \int_{\mathfrak{C}_{\rm III}(n)} \varphi(K) P_{\rm III}(Z,K)\dot{K}$$
$$= \frac{1}{V(\mathfrak{C}_{\rm III}(n-2r))} \int_{\mathfrak{C}_{\rm III}(n-2r)} \varphi\left(U_0' \begin{pmatrix} F^{(2r)} & 0 \\ 0 & K_0 \end{pmatrix} U_0\right) \frac{\det(I+Z_0\bar{Z}_0)^{a-r}}{|\det(I+Z_0\bar{K}_0)|^{2(a-r)}} \dot{K}_0.$$

The last integral represents a function continuous in $\mathfrak{B}_{\mathrm{III}}^{(n-2r)}(\cong \mathfrak{R}_{\mathrm{III}}(n-2r)\times \mathfrak{M}_{\mathrm{III}}^{(2r)})$ *and harmonic with respects to* Z_0 *in* $\mathfrak{R}_{\mathrm{III}}(n-2r)$.

(ii) *if* $K_0 \in \mathfrak{C}_{\mathrm{III}}(n)$, *then*

$$\lim_{Z\to K_0}\int_{\mathfrak{C}_{\mathrm{III}}(n)}\varphi(K)P_{\mathrm{III}}(Z,K)\dot{K}=\varphi(K_0).$$

Proof It is sufficient to prove the particular case with $r=1$ and

$$Z=\begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}\dot{+}O^{(n-2)},\quad Q=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\dot{+}O^{(n-2)},$$

that is,

$$\begin{aligned}&\lim_{\rho\to 1}\frac{1}{V(\mathfrak{C}_{\mathrm{III}}(n))}\int_{\mathfrak{C}_{\mathrm{III}}(n)}\varphi(K)\frac{(1-\rho^2)^{2a}}{|1-\rho\bar{k}_{12}|^{4a}}\dot{K}\\ =&\frac{1}{V(\mathfrak{C}_{\mathrm{III}}(n-2))}\int_{\mathfrak{C}_{\mathrm{III}}(n-2)}\varphi\left(\begin{pmatrix} F^{(2)} & O \\ 0 & K_1 \end{pmatrix}\right)\dot{K}_1.\end{aligned}\qquad (2.10.1)$$

First we consider the case n being even. Now $a=\frac{1}{2}(n-1)$. Let $K\in\mathfrak{C}_{\mathrm{III}}(n)$ and write

$$K=\begin{pmatrix} 0 & k \\ -k' & L \end{pmatrix},\quad L=-L'. \qquad (2.10.2)$$

From $K\bar{K}'=I$, we see that

$$k\bar{k}'=1,\quad k\bar{L}'=0,\quad k'\bar{k}+L\bar{L}'=I^{(n-1)}. \qquad (2.10.3)$$

Conversely, for any given vector

$$k=(k_{12},\cdots,k_{1n})$$

such that $k\bar{k}'=1$, there is a skew-symmetric matrix L satisfying (2.10.3). In fact, there exists a unitary matrix $U=U^{(n-1)}$ such that

$$k=(1,0,\cdots,0)U.$$

The matrix

$$L=U'\begin{pmatrix} 0 & 0 \\ 0 & F^{(n-2)} \end{pmatrix}U$$

satisfies our requirement.

Now we have

$$\frac{1}{V(\mathfrak{C}_{\rm III}(n))}\int_{\mathfrak{C}_{\rm III}(n)}\varphi(K)\frac{(1-\rho^2)^{n-1}}{|1-\rho\bar{k}_{12}|^{2(n-1)}}\dot{K}$$
$$=\int_{k\bar{k}'=1}\psi(k)\frac{(1-\rho^2)^{n-1}}{|1-\rho\bar{k}_{12}|^{2(n-1)}}\dot{k},$$

where

$$\psi(k)=\frac{1}{V(\mathfrak{C}_{\rm III}(n))}\int_{\substack{kL=0\\ k'k+L\bar{L}=I}}\varphi\left(\left(\begin{array}{cc}0 & k\\ -k' & L\end{array}\right)\right)\dot{L}.$$

By Theorem 1.3.2, we have

$$\lim_{\rho\to 1}\frac{1}{V(\mathfrak{C}_{\rm III}(n))}\int_{\mathfrak{C}_{\rm III}(n)}\varphi(K)\frac{(1-\rho^2)^{n-1}}{|1-\rho\bar{k}_{12}|^{2(n-1)}}\dot{K}$$
$$=\omega_{2n-3}\psi(1,0,\cdots,0)=\frac{1}{V(\mathfrak{C}_{\rm III}(n-2))}\int_{\mathfrak{C}_{\rm III}(n-2)}\varphi\left(\left(\begin{array}{cc}F^{(2)} & 0\\ 0 & L\end{array}\right)\right)\dot{L}.$$

Therefore we have (2.10.1) for even n.

Now we consider the case with odd n. The closure of $\mathfrak{R}_{\rm III}(n)$ can be imbedded into that of $\mathfrak{R}_{\rm III}(n+1)$ and $\mathfrak{C}_{\rm III}(n)$ is contained in $\mathfrak{C}_{\rm III}(n+1)$. In fact, any $U_1\in\mathfrak{C}_{\rm III}(n+1)$ can be written in the form (Hua and Look [1])

$$K_1=\left(\begin{array}{cc}K & U'h'\\ -hU & 0\end{array}\right),\quad K=U'F^{(n)}U.\quad h=(0,\cdots,0,e^{i\theta}).$$

Now, for any given function $\varphi(K)$ continuous on $\mathfrak{C}_{\rm III}(n)$, we can define a function $\varphi^*(K_1)$ continuous on $\mathfrak{C}_{\rm III}(n+1)$ such that

$$\varphi^*(K_1)=\varphi(K).$$

According to (2.9.10), we have

$$V(\mathfrak{C}_{\rm III}(n))=\frac{1}{2\pi}V(\mathfrak{C}_{\rm III}(n+1)).$$

Then

$$\frac{1}{V(\mathfrak{C}_{\rm III}(n))}\int_{\mathfrak{C}_{\rm III}(n)}\varphi(K)\frac{\det(I+Z\bar{Z})^{n/2}}{|\det(I+Z\bar{K})|^n}\dot{K}$$
$$=\frac{1}{2\pi V(\mathfrak{C}_{\rm III}(n))}\int_0^{2\pi}\int_{\mathfrak{C}_{\rm III}(n)}\varphi^*\left(\left(\begin{array}{cc}K & U'h'\\ -hU & 0\end{array}\right)\right)\frac{\det(I+Z\bar{Z})^{n/2}}{|\det(I+Z\bar{K})|^n}d\theta\dot{K}$$

$$=\frac{1}{V(\mathfrak{C}_{\rm III}(n+1))}\int_{\mathfrak{C}_{\rm III}(n+1)}\varphi^*(K_1)\frac{\det\left(I^{(n+1)}+\begin{pmatrix} Z & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} \bar Z & 0\\ 0 & 0\end{pmatrix}\right)^{n/2}}{\left|\det\left(I^{(n+1)}+\begin{pmatrix} Z & 0\\ 0 & 0\end{pmatrix}\bar K_1\right)\right|^{n}}\dot K_1.$$

Since $n+1$ is even, we can apply the result just proved to obtain the formula (2.10.1) for odd n.

§2.11. A Dirichlet problem of $\mathfrak{R}_{\rm III}$

A real-valued function $u(Z)$ is said to be harmonic on the closure of $\mathfrak{R}_{\rm III}$ if it is continuous on the characteristic manifold $\mathfrak{C}_{\rm III}(n)$, and on each $\mathfrak{B}_{\rm III}^{(n-2r)}\left(r=0,1,\cdots,\left[\frac{n}{2}\right]-1\right)$ it is harmonic with respect to the coordinate of the base space $\mathfrak{R}_{\rm III}(n-2r)$.

Similarly, the solution of the Dirichlet problem of $\mathfrak{R}_{\rm III}$ is given by

Theorem 2.11.1 Given a real-valued continuous function $\varphi(K)$ on the characteristic manifold $\mathfrak{C}_{\rm III}$, the Poisson integral

$$u(Z)=\int_{\mathfrak{C}_{\rm III}}\varphi(K)P_{\rm III}(Z,K)\dot K$$

gives the unique function which is harmonic on the closure of $\mathfrak{R}_{\rm III}$ and takes the given boundary value $\varphi(K)$ on $\mathfrak{C}_{\rm III}$.

Chapter III

The Dirichlet Problem in the Hyperbolic Space of Lie-Sphere

§3.1. Harmonic functions in $\mathfrak{R}_{\rm IV}$

Let $n\geqslant 2$ and let $\mathfrak{R}_{\rm IV}$ denote the domain

$$1+|zz'|^2-2z\bar z'>0,\quad 1-|zz'|>0 \tag{3.1.1}$$

in the space of n complex variables $z=(z_1,\cdots,z_n)$. Let $\mathfrak{C}_{\rm IV}$ denote the characteristic manifold of $\mathfrak{R}_{\rm IV}$, that is, the set of vectors

$$\xi=e^{i\theta}x,\quad xx'=1,\quad 0\leqslant\theta<\pi,$$

where x is a real n-vector. The total volume of $\mathfrak{C}_{\mathrm{IV}}$ is equal to

$$V(\mathfrak{C}_{\mathrm{IV}}) = \frac{2\pi^{\frac{n}{2}+1}}{\Gamma\left(\frac{n}{2}\right)}.$$

Since $\mathfrak{R}_{\mathrm{IV}}$ is a transitive domain, we proved previously (Look [1]) that it admits an invariant quadratic differential form

$$dzT^{\mathrm{IV}}\overline{dz'},$$

where

$$T^{\mathrm{IV}} = \frac{1}{(1+|zz'|^2-2z\overline{z}')^2}\Bigg[(1+|zz'|^2-2z\overline{z'})I^{(n)} - 2\begin{pmatrix} z \\ \overline{z} \end{pmatrix}'\begin{pmatrix} 1-2z\overline{z}' & \overline{zz}' \\ zz' & -1 \end{pmatrix}\overline{\begin{pmatrix} z \\ \overline{z} \end{pmatrix}}\Bigg]. \tag{3.1.2}$$

The inversion of T^{IV} is equal to

$$(T^{\mathrm{IV}})^{-1} = (1+|zz'|^2-2z\overline{z}')(I-2\overline{z}'z) + 2(z'-zz'\overline{z}')(\overline{z}-\overline{zz}'z). \tag{3.1.3}$$

In fact, it follows from the formal identity about matrices that

$$\begin{aligned}(I-PQ\overline{P}')^{-1} =& I + \sum_{l=1}^{\infty}(PQ\overline{P}')^l = I + P\sum_{l=0}^{\infty}(Q\overline{P}'P)^l Q\overline{P}' \\ =& I + P(I-Q\overline{P}'P)^{-1}Q\overline{P}' = I + P(Q^{-1}-\overline{P}'P)^{-1}\overline{P}'.\end{aligned}$$

Taking

$$Q = \frac{2}{1+|zz'|^2-2z\overline{z}'}\begin{pmatrix} 1-2z\overline{z}' & \overline{zz}' \\ zz' & -1 \end{pmatrix}, \quad P = \begin{pmatrix} z \\ \overline{z} \end{pmatrix}',$$

we have

$$\begin{aligned}(T^{\mathrm{IV}})^{-1} =& (1+|zz'|^2-2\overline{z}z')\Bigg\{I^{(n)} + \begin{pmatrix} z \\ \overline{z} \end{pmatrix}'\Bigg[\frac{1}{2}\begin{pmatrix} 1 & \overline{zz}' \\ zz' & -1+2z\overline{z}' \end{pmatrix} \\ & - \overline{\begin{pmatrix} z \\ \overline{z} \end{pmatrix}}\begin{pmatrix} z \\ \overline{z} \end{pmatrix}'\Bigg]^{-1}\overline{\begin{pmatrix} z \\ \overline{z} \end{pmatrix}}\Bigg\} \\ =& (1+|zz'|^2-2z\overline{z}')I + 2\begin{pmatrix} z \\ \overline{z} \end{pmatrix}'\begin{pmatrix} 1 & -\overline{zz}' \\ -zz' & -1+2z\overline{z}' \end{pmatrix}\overline{\begin{pmatrix} z \\ \overline{z} \end{pmatrix}}\end{aligned}$$

$$
\begin{aligned}
&=(1+|zz'|^2-2z\overline{z}')\left[I-2\begin{pmatrix} z \\ \overline{z}\end{pmatrix}'\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\overline{\begin{pmatrix} z \\ \overline{z}\end{pmatrix}}\right]\\
&+2\begin{pmatrix} z \\ \overline{z}\end{pmatrix}'\overline{\begin{pmatrix} 1 & -\overline{z}\overline{z}' \\ -zz' & |zz'|^2\end{pmatrix}\begin{pmatrix} z \\ \overline{z}\end{pmatrix}}\\
&=(1+|zz'|^2-2z\overline{z}')(I-2\overline{z}'z)+2(z'-\overline{z}\overline{z}'\overline{z}')(\overline{z}-\overline{z}\overline{z}'z).
\end{aligned}
$$

Consequently, we have an invariant differential operator

$$\Delta_{\rm IV}=\overline{\partial_z}(T^{\rm IV})^{-1}\partial_z',$$

where $\partial_z=\left(\frac{\partial}{\partial z_1},\cdots,\frac{\partial}{\partial z_n}\right)$ and the convention of §2.1 holds also here. More precisely,

$$
\begin{aligned}
\Delta_{\rm IV}=&(1+|zz'|^2-2z\overline{z}')\overline{\partial}_z(I-2\overline{z}'z)\partial_z'\\
&+2\overline{\partial}_z(z'-zz'\overline{z}')(\overline{z}-\overline{z}\overline{z}'z)\partial_z'\\
=&(1+|zz'|^2-2z\overline{z}')\left(\sum_{\alpha=1}^n\frac{\partial^2}{\partial z_\alpha\partial\overline{z}_\alpha}-2\sum_{\alpha,\beta=1}^n z_\alpha\overline{z}_\beta\frac{\partial^2}{\partial z_\alpha\partial\overline{z}_\beta}\right)\\
&+2\sum_{\alpha,\beta=1}^n(\overline{z}_\alpha-\overline{z}\overline{z}'z_\alpha)(z_\beta-zz'\overline{z}\beta)\frac{\partial^2}{\partial z_\alpha\partial\overline{z}_\beta}.
\end{aligned}\tag{3.1.4}
$$

A real-valued function $u(z)$ possessing continuous second derivatives is said to be harmonic in $\mathfrak{R}_{\rm IV}$ if it satisfies the partial differential equation

$$\Delta_{\rm IV}\circ u(z)=0.$$

Obviously, we have

Theorem 3.1.1 *If $u(z)$ is harmonic in $\mathfrak{R}_{\rm IV}$, the function obtained from $u(z)$ by a transformation of the group $\Gamma^{\rm IV}$ of motions of $\mathfrak{R}_{\rm IV}$ remains to be harmonic.*

It is known (Hua[2]) that the transformation of the group $\Gamma^{\rm IV}$ is of the form

$$
\begin{aligned}
w=&\left\{\left[\left(\frac{zz'+1}{2},i\frac{zz'-1}{2}\right)A'+zB'\right]\begin{pmatrix} 1 \\ i\end{pmatrix}\right\}^{-1}\\
&\times\left\{\left(\frac{zz'+1}{2},i\frac{zz'-1}{2}\right)C'+zD'\right\},
\end{aligned}\tag{3.1.5}
$$

where $A=A^{(2)},B=B^{(2,n)},C=C^{(n,2)},D=D^{(n)}$ are real matrices satisfying

$$AA'-BB'=I^{(2)},\quad CC'-DD'=-I^{(n)},\quad AC'=BD',\quad \det A>0.\tag{3.1.6}$$

The transformation (3.1.5) which carries the point $t = (t_1, \cdots, t_n)$ in $\mathfrak{R}_{IV}$ into the point $0 = (0, \cdots, 0)$ can be written as

$$w = \left\{ \left[\left(\frac{zz'+1}{2}, i\frac{zz'-1}{2} \right) - zT' \right] A' \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}^{-1} \times \left\{ z - \left(\frac{zz'+1}{2}, i\frac{zz'-1}{2} \right) T \right\} D', \tag{3.1.7}$$

where

$$AA' = (I - TT')^{-1}, \quad D'D = (I - T'T)^{-1}, \quad \det A > 0 \tag{3.1.8}$$

and

$$T = 2 \begin{pmatrix} tt'+1 & i(tt'-1) \\ \overline{t}\overline{t}'+1 & -i(\overline{t}\overline{t}'-1) \end{pmatrix}^{-1} \begin{pmatrix} t \\ \overline{t} \end{pmatrix}. \tag{3.1.9}$$

We now consider the Poisson kernel $P_{IV}(z, \xi)$ of $\mathfrak{R}_{IV}$ under the transformation of Γ^{IV}. It is known that (Hua [4])

$$P_{IV}(z, \xi) = \frac{1}{V(\mathfrak{C}_{IV})} \cdot \frac{(1 + |zz'|^2 - 2z\overline{z}')^{n/2}}{|1 + zz'\overline{\xi}\overline{\xi}' - 2z\overline{\xi}'|^n}. \tag{3.1.10}$$

We at first prove

Theorem 3.1.2 $\mathfrak{C}_{IV}$ *is invariant under the group* Γ^{IV}.

Proof Evidently $\mathfrak{C}_{IV}$ is invariant under the group of stability Γ_0^{IV}, the element of which is of the form

$$w = e^{i\psi} z\Gamma, \tag{3.1.11}$$

where Γ is an $n \times n$ real orthogonal matrix and ψ is real. Hence we can restrict ourselves to proving that $\mathfrak{C}_{IV}$ is invariant under the transformation (3.1.7) where

$$T = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \end{pmatrix}, \quad 1 > \lambda_1 \geqslant \lambda_2 \geqslant 0 \tag{3.1.12}$$

and

$$A = \begin{pmatrix} \frac{1}{\sqrt{1-\lambda_1^2}} & 0 \\ 0 & \frac{1}{\sqrt{1-\lambda_2^2}} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{\sqrt{1-\lambda_1^2}} & 0 \\ 0 & \frac{1}{\sqrt{1-\lambda_2^2}} \end{pmatrix} \dot{+} \Gamma^{(n-2)}. \tag{3.1.13}$$

Whenever $z = \xi = e^{i\theta}x$, the corresponding point of (3.1.7) is

$$w = \left\{ \left[\left(\frac{e^{2i\theta}+1}{2}, i\frac{e^{2i\theta}-1}{2} \right) - e^{i\theta}xT' \right] A' \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}^{-1}$$

$$\times\left\{e^{i\theta}x-\left(\frac{e^{2i\theta}+1}{2},i\frac{e^{2i\theta}-1}{2}\right)T\right\}D'$$

$$=\left\{[(\cos\theta,-\sin\theta)-(\lambda_1x_1,\lambda_2x_2)]\begin{pmatrix}\dfrac{1}{\sqrt{1-\lambda_1^2}}\\ \dfrac{i}{\sqrt{1-\lambda_2^2}}\end{pmatrix}\right\}^{-1}$$

$$\times\{x-(\lambda_1\cos\theta,-\lambda_2\sin\theta,0,\cdots,0)\}D'$$

$$=\left\{\frac{\cos\theta-\lambda_1x_1}{\sqrt{1-\lambda_1^2}}-i\frac{\sin\theta-\lambda_2x_2}{\sqrt{1-\lambda_2^2}}\right\}^{-1}$$

$$\times\left(\frac{x_1-\lambda_1\cos\theta}{\sqrt{1-\lambda_1^2}},\frac{x_2+\lambda_2\sin\theta}{\sqrt{1-\lambda_2^2}},x_3,\cdots,x_n\right)$$

$$=e^{i\psi}y,$$

where

$$\psi=\arg\left\{\frac{\cos\theta-\lambda_1x_1}{\sqrt{1-\lambda_1^2}}-i\frac{\sin\theta+\lambda_1x_2}{\sqrt{1-\lambda_1^2}}\right\}$$

and

$$y=\left|\frac{\cos\theta-\lambda_1x_1}{\sqrt{1-\lambda_1^2}}-i\frac{\sin\theta+\lambda_2x_2}{\sqrt{1-\lambda_2^2}}\right|^{-1}\left(\frac{x_1-\lambda_1\cos\theta}{\sqrt{1-\lambda_1^2}},\frac{x_2+\lambda_2\sin\theta}{\sqrt{1-\lambda_2^2}},x_3,\cdots,x_n\right).$$

Obviously, y is a real vector. It remains to prove $yy'=1$. In fact,

$$yy'=\frac{\dfrac{(x_1-\lambda_1\cos\theta)^2}{1-\lambda_1^2}+\dfrac{(x_2+\lambda_2\sin\theta)^2}{1-\lambda_2^2}+x_3^2+\cdots+x_n^2}{\dfrac{(\cos\theta-\lambda_1x_1)^2}{1-\lambda_1^2}+\dfrac{(\sin\theta+\lambda_2x_2)^2}{1-\lambda_2^2}}$$

$$=\{(1-\lambda_2^2)[(x_1-\lambda_1\cos\theta)^2-(1-\lambda_1^2)x_1^2]$$

$$+(1-\lambda_1^2)[(x_2+\lambda_2\sin\theta)^2-(1-\lambda_2^2)x_2]+(1-\lambda_1^2)(1-\lambda_2^2)\}$$

$$\times\{(1-\lambda_2^2)(\cos\theta-\lambda_1x_1)^2+(1-\lambda_1^2)(\sin\theta+\lambda_2x_2)^2\}^{-1}=1.$$

The theorem is proved.

Theorem 3.1.3 *After the transformation* (3.1.7), *the Poisson kernel becomes*

$$P_{\mathrm{IV}}(w,\xi)=\frac{|1+tt'\overline{\xi\xi}'-2t\overline{\xi}'|^n}{(1+|tt'|^2-2t\overline{t}')^{n/2}}P_{\mathrm{IV}}(z,\xi),$$

where $\zeta\in\mathfrak{C}_{\mathrm{IV}}$ *is the point corresponding to the point* $\xi\in\mathfrak{C}_{\mathrm{IV}}$ *under* (3.1.7).

Proof The Cauchy kernel $H_{\rm IV}(z,\overline{\xi})$ of $\mathfrak{R}_{\rm IV}$ is known to be (Hua[4])

$$H_{\rm IV}(z,\overline{\xi}) = \frac{1}{V(\mathfrak{C}_{\rm IV})} \cdot \frac{1}{(1+zz'\overline{\xi\xi}' - 2z\overline{\xi}')^{n/2}}.$$

Hence the Poisson kernel and the Cauchy kernel satisfy the following relation

$$P_{\rm IV}(z,\overline{\xi}) = \frac{H(z,\overline{\xi})H(\xi,\overline{z})}{H(z,\overline{z})}. \tag{3.1.14}$$

It is known that, after the transformation (3.1.7), the Cauchy kernel suffers

$$H_{\rm IV}(z,\overline{\xi}) = H_{\rm IV}(w,\overline{\zeta})B^{\frac{1}{2}}(z,t)\overline{B^{\frac{1}{2}}(\xi,t)}, \tag{3.1.15}$$

where $B(z,t)$ is the functional determinant of (3.1.7). Hence,

$$P_{\rm IV}(z,\xi) = P_{\rm IV}(w,\zeta)|B(\xi,t)|. \tag{3.1.16}$$

If we take $z=t$ in (3.1.15), then the corresponding point is $w=0$, and (3.1.15) becomes

$$H_{\rm IV}(t,\overline{\xi}) = H_{\rm IV}(0,\overline{\zeta})B^{\frac{1}{2}}(t,t)\overline{B^{\frac{1}{2}}(\xi,t)}. \tag{3.1.17}$$

It is known (Hua[4]) that

$$B(t,t) = \frac{1}{(1+|tt'|^2 - 2t\overline{t}')^{n/2}}$$

and $H_{\rm IV}(0,\overline{\zeta}) = \dfrac{1}{V(\mathfrak{C}_{\rm IV})}$. We obtain from (3.1.17)

$$|B(\xi,t)| = \frac{(1+|tt'|^2 - 2t\overline{t}')^{n/2}}{|1+tt'\overline{\xi\xi}' - 2t\overline{\xi}'|^n}.$$

Substituting the above value into (3.1.16), we get the required result.

Theorem 3.1.4 *The Poisson kernel $P_{\rm IV}(z,\overline{\xi})$ is harmonic in $\mathfrak{R}_{\rm IV}$ with respect to the variable z.*

Proof After Theorems 3.1.1 and 3.1.3, it is sufficient to prove that

$$[\Delta_{\rm IV} \circ P_{\rm IV}(z,\xi)]_{z=0} = 0.$$

In fact, by (3.1.4), we have

$$[\Delta_{\rm IV} \circ P_{\rm IV}(z,\xi)]_{z=0} = \left[\sum_{\alpha=1}^{n} \frac{\partial^2 P_{\rm IV}(z,\xi)}{\partial z_\alpha \partial \overline{z}_\alpha}\right]_{z=0}$$

$$=\frac{1}{V(\mathfrak{C}_{\mathrm{IV}})}\left[\sum_{\alpha=1}^{n}\frac{\partial^2}{\partial z_\alpha\partial\overline{z}_\alpha}(1+|zz'|^2-2z\overline{z}')^{n/2}(1+zz'\overline{\xi\xi}'-2z\overline{\xi}')^{-n/2}\right.$$
$$\left.\times(1+\overline{zz}'\xi\xi'-2\overline{z}\xi')^{-n/2}\right]_{z=0}$$
$$=\frac{1}{V(\mathfrak{C}_{\mathrm{IV}})}\left[\sum_{\alpha=1}^{n}\frac{\partial^2}{\partial z_\alpha\partial\overline{z}_\alpha}(1-nz\overline{z}'+\cdots)(1+nz\overline{\xi}'+\cdots)\right.$$
$$\left.\times(1+n\overline{z}\xi'+\cdots)\right]_{z=0}$$
$$=\frac{1}{V(\mathfrak{C}_{\mathrm{IV}})}\left[-n\sum_{\alpha=1}^{n}1+n^2\sum_{\alpha=1}^{n}\xi_\alpha\overline{\xi}_\alpha\right]=\frac{1}{V(\mathfrak{C}_{\mathrm{IV}})}[-n^2+n^2xx']=0.$$

This proves the theorem.

§3.2. The boundary properties

For the sake of convenience, we take a linear transformation

$$\overset{*}{z}_1=z_1-iz_2,\quad \overset{*}{z}_2=z_1+iz_2,\quad \overset{*}{z}_\alpha=z_\alpha\quad(\alpha=3,\cdots,n),\tag{3.2.1}$$

which carries $\mathfrak{R}_{\mathrm{IV}}$ onto $\mathfrak{R}^*_{\mathrm{IV}}$, the domain defined by

$$\begin{aligned}&1+|\overset{*}{z}_1\overset{*}{z}_2+\overset{*}{z}{}_3^2+\cdots+\overset{*}{z}{}_n^2|^2-[|\overset{*}{z}_1|^2+|\overset{*}{z}_2|^2+2(|\overset{*}{z}_3|^2+\cdots+|\overset{*}{z}_n|^2)]>0,\\&1-|\overset{*}{z}_1\overset{*}{z}_2+\overset{*}{z}{}_3^2+\cdots+\overset{*}{z}{}_n^2|>0.\end{aligned}\tag{3.2.2}$$

Denote by $\overline{\mathfrak{R}}^*_{\mathrm{IV}}$ the closure of $\mathfrak{R}^*_{\mathrm{IV}}$ and by $\mathfrak{B}^*_{\mathrm{IV}}$ its boundary. The characteristic manifold $\mathfrak{C}_{\mathrm{IV}}$ is transformed to be $\mathfrak{C}^*_{\mathrm{IV}}$, the points of which can be represented as

$$\overset{*}{\xi}=e^{i\theta}(x_1-ix_2),\quad \overset{*}{\xi}_2=e^{i\theta}(x_1+ix_2),\quad \overset{*}{\xi}_\alpha=e^{i\theta}x_\alpha,\quad(\alpha=3,\cdots,n),\tag{3.2.3}$$

where θ and $x=(x_1,\cdots,x_n)$ are real numbers with $xx'=1$.

Let $\overset{*}{\Gamma}{}^{\mathrm{IV}}$ and $\overset{*}{\Gamma}{}_0^{\mathrm{IV}}$ be the groups corresponding to the groups Γ^{IV} and Γ_0^{IV} respectively. Since any homeomorphic mapping carries the boundary into boundary and $\mathfrak{C}^*_{\mathrm{IV}}$ is invariant under $\overset{*}{\Gamma}{}^{\mathrm{IV}}$ by Theorem 3.1.2, obviously $\mathfrak{B}^*_{\mathrm{IV}}-\mathfrak{C}^*_{\mathrm{IV}}$ is invariant under $\overset{*}{\Gamma}{}^{\mathrm{IV}}$. Moreover, $\mathfrak{B}^*_{\mathrm{IV}}-\mathfrak{C}^*_{\mathrm{IV}}$ is transitive under $\overset{*}{\Gamma}{}^{\mathrm{IV}}$, i.e.,

Theorem 3.2.1 *Any point of* $\mathfrak{B}^*_{\mathrm{IV}}-\mathfrak{C}^*_{\mathrm{IV}}$ *can be transformed into the point* $(0,1,0,\cdots,0)$ *by* $\overset{*}{\Gamma}{}^{\mathrm{IV}}$.

Proof Let t_0 be an arbitrary point of $\mathfrak{B}^*_{\mathrm{IV}}-\mathfrak{C}^*_{\mathrm{IV}}$. According to the Example 3 given is §1.1, we can assume without loss of generality that

$$t_0=(t_1,1,0,\cdots,0),\quad |t_1|<1.$$

From (3.1.7) and (3.1.8), we know that the transformation of $\overset{*}{\Gamma}^{\mathrm{IV}}$ is of the form

$$\overset{*}{w}=\left\{\left[\left(\frac{1}{2}(\overset{*}{z}Q^{-1}Q'^{-1}\overset{*}{z}'+1),\frac{i}{2}(\overset{*}{z}Q^{-1}Q'^{-1}\overset{*}{z}'-1)\right)-\overset{*}{z}Q^{-1}T'\right]A'\begin{pmatrix}1\\ i\end{pmatrix}\right\}^{-1}$$
$$\times\left\{\overset{*}{z}Q^{-1}-\left(\frac{1}{2}(\overset{*}{z}Q^{-1}Q'^{-1}\overset{*}{z}'+1),\frac{i}{2}(\overset{*}{z}Q^{-1}Q'^{-1}\overset{*}{z}'-1)\right)T\right\}D'Q, \quad (3.2.4)$$

where $Q=\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\dot{+}I^{(n-2)}$ and

$$T=2\begin{pmatrix}\overset{*}{t}Q^{-1}Q'^{-1}\overset{*}{t}'+1 & i(\overset{*}{t}Q^{-1}Q'^{-1}\overset{*}{t}'-1)\\ \overline{\overset{*}{t}Q^{-1}Q'^{-1}\overset{*}{t}'+1} & -i\overline{(\overset{*}{t}Q^{-1}Q'^{-1}\overset{*}{t}'-1)}\end{pmatrix}^{-1}\begin{pmatrix}\overset{*}{t}Q^{-1}\\ \overline{\overset{*}{t}Q^{-1}}\end{pmatrix} \quad (3.2.5)$$

with $A'A=(I-TT')^{-1}, D'D=(I-T'T)^{-1}, \det A>0$.

Now we take $\overset{*}{t}=(t_1,0,\cdots,0)$. Then

$$T=\frac{1}{2}\begin{pmatrix}t_1+\bar{t}_1 & it_1-i\bar{t}_1 & 0 & \cdots & 0\\ it_1-i\bar{t}_1 & -t_1-\bar{t}_1 & 0 & \cdots & 0\end{pmatrix}.$$

Hence

$$TT'=\begin{pmatrix}|t_1|^2 & 0\\ 0 & |t_1|^2\end{pmatrix}$$

and

$$T'T=\begin{pmatrix}|t_1|^2 & 0\\ 0 & |t_1|^2\end{pmatrix}\dot{+}O^{(n-2)}.$$

Besides we choose

$$A=\begin{pmatrix}\frac{1}{\sqrt{1-|t_1|^2}} & 0\\ 0 & \frac{1}{\sqrt{1-|t_1|^2}}\end{pmatrix} \text{ and } D=\begin{pmatrix}\frac{1}{\sqrt{1-|t_1|^2}} & 0\\ 0 & \frac{1}{\sqrt{1-|t_1|^2}}\end{pmatrix}\dot{+}I^{(n-2)}.$$

After T, A and D are so chosen, we see that the transformation (3.2.4) at the point $\overset{*}{z}=t_0$ is equal to

$$\overset{*}{w}=\left\{\left[\left(\frac{t_1+1}{2},i\frac{t_1-1}{2}\right)-\frac{1}{2}(t_1+1,it_1-i,0,\cdots,0)T'\right]A'\begin{pmatrix}1\\ i\end{pmatrix}\right\}^{-1}$$
$$\times\left\{\frac{1}{2}(t_1+1,it_1-i,0,\cdots,0)-\frac{1}{2}(t_1+1,it_1-i)T\right\}D'Q$$

$$=\left\{\left[(t_1+1, it_1-i)-\frac{1}{2}((t_1+1)(t_1+\bar{t}_1)\right.\right.$$

$$\left.\left.-(t_1-1)(t_1-\bar{t}_1), i(t_1+1)(t_1-\bar{t}_1)-i(t_1-1)(t_1+\bar{t}_1))\right]A'\begin{pmatrix}1\\ i\end{pmatrix}\right\}^{-1}$$

$$\times\left\{(t_1+1, it_1-i, 0, \cdots, 0)-\frac{1}{2}((t_1+1)(t_1+\bar{t}_1)\right.$$

$$-(t_1-1)(t_1-\bar{t}_1), i(t_1+1)(t_1-\bar{t}_1)$$

$$\left.-i(t_1-1)(t_1+\bar{t}_1), 0, \cdots, 0)\right\}D'Q$$

$$=\left\{\frac{1}{\sqrt{1-|t_1|^2}}(1-|t_1|^2, -i+i|t_1|^2)\begin{pmatrix}1\\ i\end{pmatrix}\right\}^{-1}$$

$$\times\left\{\frac{1}{\sqrt{1-|t_1|^2}}(1-|t_1|^2, -i+i|t_1|^2, 0, \cdots, 0)Q\right\}$$

$$=(0, 1, 0, \cdots, 0).$$

The theorem is proved.

By Example 3 in §1.1, we have

Theorem 3.2.2 *$\overline{\mathfrak{R}}^*_{\mathrm{IV}} \supset \mathfrak{C}^*_{\mathrm{IV}}$ form a chain of slit spaces. The slit of $\mathfrak{B}^*_{\mathrm{IV}}$ is $\mathfrak{C}^*_{\mathrm{IV}}$ and $\mathfrak{B}^*_{\mathrm{IV}}-\mathfrak{C}^*_{\mathrm{IV}}$ is homeomorphic to the topological product of $X=\{|t|<1\}$ and $Y=O(n)/O(n-2)$.*

Now we consider the Poisson integral of $\mathfrak{R}^*_{\mathrm{IV}}$

$$u(\overset{*}{z})=\int_{\mathfrak{C}^*_{\mathrm{IV}}}\varphi(\overset{*}{\xi})P^*_{\mathrm{IV}}(\overset{*}{z},\overset{*}{\xi})\,\overset{*}{\xi}, \tag{3.2.6}$$

where

$$P^*_{\mathrm{IV}}(\overset{*}{z},\overset{*}{\xi})=\frac{1}{V(\mathfrak{C}^*_{\mathrm{IV}})}[1+|\overset{*}{z}_1\overset{*}{z}_2+\overset{*}{z}{}^2_3+\cdots+\overset{*}{z}{}^2_n|^2$$
$$-(|\overset{*}{z}_1|^2+|\overset{*}{z}_2|^2+2|\overset{*}{z}_3|^2+\cdots+2|\overset{*}{z}_n|^2)]^{n/2}\times|1+\{(\overset{*}{z}_1\overset{*}{z}_2+\overset{*}{z}{}^2_3$$
$$+\cdots+\overset{*}{z}{}^2_n)(\overline{\overset{*}{\xi}_1}\,\overline{\overset{*}{\xi}_2}+\overline{\overset{*}{\xi}{}^2_3}+\cdots+\overline{\overset{*}{\xi}{}^2_n})$$
$$-(\overset{*}{z}_1\overline{\overset{*}{\xi}_1}+\overset{*}{z}_2\overline{\overset{*}{\xi}_2}+2\overset{*}{z}_3\overline{\overset{*}{\xi}_3}+\cdots+2\overset{*}{z}_n\overline{\overset{*}{\xi}_n})|\}^{-n}$$

and

$$V(\mathfrak{C}^*_{\mathrm{IV}})=\frac{2\pi^{\frac{n}{2}+1}}{\Gamma\left(\frac{n}{2}\right)}. \tag{3.2.7}$$

Theorem 3.2.3 *Let $\varphi(\overset{*}{\xi})$ be a real-valued function continuous in the characteristic manifold $\mathfrak{C}^*_{\mathrm{IV}}$ of $\mathfrak{R}^*_{\mathrm{IV}}$.*

(i) *If* $z_0 = (t_1, 1, 0, \cdots, 0)$ A *is a point of* $\mathfrak{B}_{\mathrm{IV}}^* - \mathfrak{C}_{\mathrm{IV}}^* (A \in \overset{*}{\Gamma}{}_0^{\mathrm{IV}})$, *then*

$$\lim_{\overset{*}{z} \to z_0} \int_{\mathfrak{C}_{\mathrm{IV}}^*} \varphi(\overset{*}{\xi}) P_{\mathrm{IV}}^*(\overset{*}{z}, \overset{*}{\xi}) \dot{\overset{*}{\xi}} = \frac{1}{\pi} \int_0^{\pi} \varphi((e^{2i\theta}, 1, 0, \cdots, 0)A) \frac{1 - |t_1|^2}{|1 - t_1 e^{-2i\theta}|^2} d\theta. \tag{3.2.8}$$

Notice that the last integral is a harmonic function in the usual sense with respect to the variable t_1.

(ii) *If* $\overset{*}{\zeta} \in \mathfrak{C}_{\mathrm{IV}}^*$, *then*

$$\lim_{\overset{*}{z} \to \overset{*}{\zeta}} \int_{\mathfrak{C}_{\mathrm{IV}}^*} \varphi(\overset{*}{\xi}) P_{\mathrm{IV}}^*(\overset{*}{z}, \overset{*}{\xi}) \dot{\overset{*}{\xi}} = \varphi(\overset{*}{\zeta}).$$

Proof By a method previously used, we should only prove that when $z = (0, r, 0, \cdots, 0), 0 \leqslant r < 1$, and $z_0 = (0, 1, 0, \cdots, 0)$ we have

$$\lim_{r \to 1} \int_{\mathfrak{C}_{\mathrm{IV}}^*} \varphi(\overset{*}{\xi}) P_{\mathrm{IV}}^*((0, r, 0, \cdots, 0), \overset{*}{\xi}) \dot{\overset{*}{\xi}} = \frac{1}{\pi} \int_0^{\pi} \varphi(e^{2i\theta}, 1, 0, \cdots, 0) d\theta. \tag{3.2.9}$$

In fact,

$$\int_{\mathfrak{C}_{\mathrm{IV}}^*} \varphi(\overset{*}{\xi}) P_{\mathrm{IV}}^*((0, r, 0, \cdots, 0), \overset{*}{\xi}) \dot{\overset{*}{\xi}} = \frac{1}{V(\mathfrak{C}_{\mathrm{IV}}^*)} \int_{xx'=1} \int_0^{\pi} \varphi(\overset{*}{\xi}) \frac{(1 - r^2)^{n/2}}{|1 - re^{-i\theta}(x_1 - ix_2)|^n} d\theta \dot{x}.$$

We make the following change:

$$y_1 - iy_2 = e^{-i\theta}(x_1 - ix_2), y_3 = x_3, \cdots, y_n = x_n, \theta = \theta.$$

Then

$$\int_{\mathfrak{C}_{\mathrm{IV}}^*} \varphi(\overset{*}{\xi}) P_{\mathrm{IV}}^*((0, r, 0, \cdots, 0), \overset{*}{\xi}) \dot{\overset{*}{\xi}} = \frac{1}{V(\mathfrak{C}_{\mathrm{IV}}^*)} \int_{yy'=1} \psi(y) \frac{(1 - r^2)^{n/2}}{|1 - r(y_1 - iy_2)|^n} \dot{y},$$

where

$$\psi(y) = \int_0^{\pi} \varphi(e^{2i\theta}(y_1 - iy_2), y_1 + iy_2, e^{i\theta} y_3, \cdots, e^{i\theta} y_n) d\theta.$$

Applying Theorems 1.3.2, we have

$$\lim_{r \to 1} \frac{1}{V(\mathfrak{C}_{\mathrm{IV}}^*)} \int_{yy'=1} \psi(y) \frac{(1 - r^2)^{n/2}}{|1 - r(y_1 - iy_2)|^n} \dot{y} = \frac{\omega_{n-1}}{V(\mathfrak{C}_{\mathrm{IV}}^*)} \psi(1, 0, \cdots, 0).$$

This proves formula (3.2.9).

§3.3. A Dirichlet problem of $\mathfrak{R}_{\mathrm{IV}}$

A real-valued function $\varphi(\xi)$ harmonic in $\mathfrak{R}_{\mathrm{IV}}$ is said to be harmonic on the closure of $\mathfrak{R}_{\mathrm{IV}}$, if it is continuous in $\overline{\mathfrak{R}}_{\mathrm{IV}}$ and on $\mathfrak{B}_{\mathrm{IV}} - \mathfrak{C}_{\mathrm{IV}}$; it is harmonic in the usual sense with respect to the coordinate of $X = \{|t| < 1\}$.

Applying Theorems 3.2.3 and 1.2.2 we solve the corresponding Dirichlet problem of $\mathfrak{R}_{\mathrm{IV}}$:

Theorem 3.3.1 *If a real-valued continuous function $\varphi(\xi)$ is given in the characteristic manifold $\mathfrak{C}_{\mathrm{IV}}$ of $\mathfrak{R}_{\mathrm{IV}}$, then*

$$u(z) = \int_{\mathfrak{C}_{\mathrm{IV}}} \varphi(\xi) P_{\mathrm{IV}}(z,\xi)\dot{\xi}$$

is the unique function harmonic on the closure of $\mathfrak{R}_{\mathrm{IV}}$, which takes the given boundary value $\varphi(\xi)$ on $\mathfrak{C}_{\mathrm{IV}}$.

Chapter IV

APPLICATIONS

In this chapter we give a few applications and remarks which are not too lengthy to be included here.

§4.1. A convergence theorem in the theory of representations

Let $\mathfrak{U}(n)$ be the unitary group of order n and let $A_{f_1,\cdots,f_n}(U)$ defined for $U \in \mathfrak{U}(n)$ be the (unitary) representation with the signature $(f_1, f_2, \cdots, f_n)$, where $f_1, \cdots, f_n$ are integers satisfying $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n$ Sometimes, for simplicity, we use f to denote $(f_1, f_2, \cdots, f_n)$. Let $N(f)$ be the order of the matrix $A_f(U)$ and let

$$A_f(U) = (a_{ij}^f(U))_{1 \leqslant i,j \leqslant N(f)}.$$

After normalization, we let

$$\varphi_{ij}^f(U) = \sqrt{\frac{N(f)}{C}}\, a_{ij}^f(U), \tag{4.1.1}$$

where $C = V(\mathfrak{C}_{\mathrm{I}}(n,n))$ is the total volume of $\mathfrak{U}(n) (= \mathfrak{C}_{\mathrm{I}}(n,n))$ (see §2.2).

We put, in Theorem 2.4.2(ii), $m = n$ and $Z = rV$, and have

$$\lim_{r\to 1} \frac{1}{C} \int_{\mathfrak{U}(n)} \varphi(U) \frac{(1-r^2)^{n^2}}{|\det(I - rV\overline{U}')|^n} \dot{U} = \varphi(V). \tag{4.1.2}$$

It is known (Hua [4]) that

$$\begin{aligned}\det(I - Z\overline{U}')^{-n} =& C\sum_{f_1\geqslant\cdots\geqslant f_n\geqslant 0}\sum_{i,j=1}^{N(f)} \varphi^f_{ij}(Z)\overline{\varphi^f_{ij}(U)}\\ =& \sum_{f_1\geqslant\cdots\geqslant f_n\geqslant 0} N(f)\mathrm{tr}(A_f(Z\overline{U}')),\end{aligned} \tag{4.1.3}$$

which converges uniformly in any compact subset of $\mathfrak{R}_{\mathrm{I}}$, in particular, in the closed set

$$rI - Z\overline{Z}' \geqslant 0, \quad 0 \leqslant r < 1.$$

The expression

$$\varphi^f_{ij}(U)\overline{\varphi^g_{kl}(U)}$$

with $f_1 \geqslant \cdots \geqslant f_n \geqslant 0$ and $g_1 \geqslant \cdots \geqslant g_n \geqslant 0$ appears in the representation

$$A_f(U) \times \overline{A_g(U)}.$$

Therefore, $\varphi^f_{ij}(U) \times \overline{\varphi^g_{kl}(U)}$ can be expressed as a finite linear combination of the functions in the sequence (4.1.1). More precisely,

$$a^f_{ij}(U)\overline{a^g_{kl}(U)} = \sum \lambda^h_{st} a^h_{st}(U)$$

with

$$\sum |\lambda^h_{pq}|^2 = 1.$$

Then the Poisson kernel of $\mathfrak{R}_{\mathrm{I}}(n,n)$ equals

$$\begin{aligned}&\frac{1}{C}\frac{\det(I - Z\overline{Z}')^n}{|\det(I - Z\overline{U}')|^{2n}}\\ =& C\det(I - Z\overline{Z}')^n \sum_{f\geqslant 0}\sum_{g\geqslant 0}\sum_{i,j}\sum_{s,t} \varphi^f_{ij}(Z)\overline{\varphi^f_{ij}(U)}\varphi^g_{st}(U)\overline{\varphi^g_{st}(Z)}\\ =& \sum_h\sum_{k,l} \Phi^h_{kl}(Z)\overline{\varphi^h_{kl}(U)}.\end{aligned} \tag{4.1.4}$$

This series converges uniformly in $rI - Z\overline{Z}' \geqslant 0$.

Multiplying (4.1.4) by $\varphi^f_{ij}(U)$ on both sides and integrating term by term, we obtain

$$\Phi^f_{ij}(Z) = \frac{1}{C}\int_{\mathfrak{U}(n)} \varphi^f_{ij}(U)\frac{\det(I - Z\overline{Z}')^n}{|\det(I - Z\overline{U}')|^{2n}}\dot{U}, \tag{4.1.5}$$

which by (4.1.2) becomes

$$\lim_{r\to 1} \Phi^f_{ij}(rU) = \varphi^f_{ij}(U). \tag{4.1.6}$$

Now, let $u(U)$ be an arbitrary continuous function defined in $\mathfrak{A}(n)$. We have formally the Fourier series of $u(U)$:

$$\sum_f \sum_{i,j} a^f_{ij}\varphi^f_{ij}(U), \quad a^f_{ij} = \int_{\mathfrak{A}(n)} u(U)\overline{\varphi^f_{ij}(U)}\dot{U}.$$

Since

$$\begin{aligned}\frac{1}{C}\int_{\mathfrak{A}} u(U)\frac{\det(I-Z\overline{Z}')^n}{|\det(I-Z\overline{U}')|^{2n}}\dot{U} &= \sum_f\sum_{i,j}\Phi^f_{ij}(Z)\int_{\mathfrak{A}} u(U)\overline{\varphi^f_{ij}(U)}\dot{U}\\ &= \sum_f\sum_{i,j} a^f_{ij}\Phi^f_{ij}(Z),\end{aligned}$$

which converges uniformly on any compact subset of $\mathfrak{R}_{\mathrm{I}}$ and

$$u(U) = \lim_{r\to 1}\left(\sum_f\sum_{i,j} a^f_{ij}\Phi^f_{ij}(rU)\right), \tag{4.1.7}$$

the right hand side is defined to be the Abel sum of the Fourier series $\sum\limits_f\sum\limits_{i,j} a^f_{ij}\varphi^f_{ij}(U)$.

Theorem 4.1.1 *Every continuous function on $\mathfrak{A}(n)$ can be represented by its Fourier series in the sense of the Abel summability.*

We shall express $\Phi^f_{ij}(rU)$ in terms of $\varphi^f_{ij}(U)$.

Let

$$B_f(rV) = \sqrt{\frac{C}{N(f)}}(\Phi^f_{ij}(rV))_{1\leqslant i,j\leqslant N(f)}. \tag{4.1.8}$$

By (4.1.5) we have

$$B_f(rV) = \frac{1}{C}\int_{\mathfrak{A}}\frac{(1-r^2)^{n^2}}{|\det(I-rV\overline{U}')|^{2n}}A_f(U)\dot{U}.$$

Let W be any unitary matrix. If we change V and U into WV and WU respectively in the above integral, then we have

$$B_f(rWV) = \frac{1}{C}\int_{\mathfrak{A}}\frac{(1-r^2)^{n^2}}{|\det(I-rV\overline{U}')|^{2n}}A_f(WU)\dot{U} = A_f(W)B_f(rV). \tag{4.1.9}$$

Similarly,

$$B_f(rVW) = B_f(rV)A_f(W). \tag{4.1.10}$$

From (4.1.9) and (4.1.10) with $V = I$, we have, for any unitary W,

$$A_f(W)B_f(rI) = B_f(rI)A_f(W).$$

By Schur's lemma, we have immediately

$$B_f(rI) = \rho^f(r)A_f(W),$$

where $\rho^f(r)$ is a function in r alone and

$$\rho^f(r) \to 1 \text{ for } r \to 1. \tag{4.1.11}$$

According to Theorem 4.1.1, we have

Theorem 4.1.2 *Let $u(U)$ be a function continuous on $\mathfrak{A}(n)$. For any given $\varepsilon > 0$, there is a number $\delta > 0$ and a positive integer N_0 such that whenever $1 - \delta < r < 1$, we have*

$$\left| u(U) - \sum_{N_0 \geqslant f_1 \geqslant \cdots \geqslant f_n \geqslant -N_0} \sum_{i,j} a_{ij}^f \rho^f(r) \varphi_{ij}^f(U) \right| < \varepsilon.$$

To guarantee that the Poisson integral

$$\int_{\mathfrak{A}} \varphi(U) P_{\mathrm{I}}(Z, U) \dot{U} \tag{4.1.12}$$

represents a harmonic function in $\mathfrak{R}_{\mathrm{I}}$, it is not necessary to assume that $\varphi(U)$ is continuous. In fact, if we assume that $\varphi(U)$ is integrable (or belongs to L), the integral (4.1.12) still exists in the interior of $\mathfrak{R}_{\mathrm{I}}$ and it has also the expansion

$$\sum_f \sum_{i,j} a_{ij}^f \Phi_{ij}^f(Z), \tag{4.1.13}$$

which converges uniformly for $rI - Z\overline{Z}' \geqslant 0$, where $0 \leqslant r < 1$. It is to be remarked that the assumption about the continuity can be replaced by a local one, i.e., if $\varphi(U)$ is a function continuous at a point U_0, then the Fouries series

$$\sum_f \sum_{i,j} a_{ij}^f \varphi_{ij}^f(U)$$

is Abel summable to $u(U_0)$.

§4.2. The distribution on unitary group

To each function which is harmonic in $\mathfrak{R}_{\rm I}$ (also in $\mathfrak{R}_{\rm II}, \mathfrak{R}_{\rm III}$ and $\mathfrak{R}_{\rm IV}$), we define a distribution. More precisely, if we start with a harmonic function

$$u(Z) = \sum_{f_1 \geqslant \cdots \geqslant f_n} \sum_{i,j=1}^{N(f)} a_{ij}^f \Phi_{ij}^f(Z),$$

the distribution is defined by the formal power series

$$u(U) = \sum_{f_1 \geqslant \cdots \geqslant f_n} \sum_{i,j} a_{ij}^f \varphi_{ij}^f(U),$$

which may converge or may not.

Given two distributions $u(U), v(U)$, we define a convolution

$$(u(U), \overline{v(U)}) = \sum_{f_1 \geqslant \cdots \geqslant f_n} \sum_{i,j} a_{ij}^f \bar{b}_{ij}^f$$

if the series converges or is summable in the Abel sense, i.e.,

$$\lim_{r\to 1} \int_{\mathfrak{C}_{\rm I}} u(U)\overline{v(rU)}\dot{U} = \lim_{r \to 1}\left[\sum_f \sum_{i,j} a_{ij}^f \bar{b}_{ij}^f \rho^f(r)\right].$$

The distribution expressed by the Poisson kernel $P_{\rm I}(rV, U)$ is called the delta function on the unitary group and is denoted by $\delta_V(U)$. We have

$$(u(U), \delta_V(U)) = \lim_{r\to 1} \int_{\mathfrak{C}_{\rm I}} u(U) P_{\rm I}(rV, U)\dot{U} = u(V).$$

The detail of the study of the theory of distribution on a compact group and on the homogeneous space will not be given here.

§4.3. A note to the harmonic functions of real variables

Let $\mathfrak{R}(m, n)$ be the set of all real $m \times n$ matrices $X = (x_{i\alpha})_{1\leqslant i\leqslant m, 1\leqslant \alpha\leqslant n}$ satisfying

$$I - XX' > 0. \tag{4.3.1}$$

Without loss of generality, we always assume $m \leqslant n$.

$\mathfrak{R} = \mathfrak{R}(m, n)$ admits a group $\mathfrak{G}$ of motions, the element of which is the transformation

$$Y = (AX + B)(CX + D)^{-1}, \tag{4.3.2}$$

where $A = A^{(m)}, B = B^{(m,n)}, C = C^{(n,m)}, D = D^{(n)}$ are real matrices satisfying

$$A'A - C'C = I^{(m)}, \quad A'B = C'D, \quad D'D - B'B = I^{(n)}. \tag{4.3.3}$$

It is not hard to see that $CX + D$ is non-singular whenever X belongs to the closure of $\mathfrak{R}$, and

$$(AX + B)(CX + D)^{-1} = (XB' + A')^{-1}(XD' + C'). \tag{4.3.4}$$

Moreover, $\mathfrak{R}$ is transitive under $\mathfrak{g}$, since for any point $X_0 \in \mathfrak{R}$ there is a transformation of $\mathfrak{g}$

$$Y = A(X - X_0)(I - X_0'X)^{-1}D^{-1} \tag{4.3.5}$$

with

$$A'A = (I - X_0X_0')^{-1} \quad D'D = (I - X_0'X_0)^{-1}, \tag{4.3.6}$$

which carries X_0 into O.

In $\mathfrak{R}$, we can introduce a Riemann metric

$$ds^2 = \operatorname{tr}(dX(I - X'X)^{-1}dX'(I - XX')^{-1}), \tag{4.3.7}$$

which is invariant under $\mathfrak{g}$. If we arrange the pairs of indices $(i\alpha)$ into the order

$$(11), (12), \cdots, (1n), (21), (22), \cdots, (2n), \cdots, (m1), (m2), \cdots, (mn),$$

then the contravariant tensor $\mathfrak{G}^{(i\alpha)(j\beta)}$ associated to the fundamental tensor $\mathfrak{G}_{(i\alpha)(j\beta)}$ of the Riemann metric (4.3.7) can be written as

$$\mathfrak{G}^{(i\alpha)(j\beta)} = \left(\delta_{ij} - \sum_{r=1}^{n} x_{i\gamma}x_{j\gamma}\right)\left(\delta_{\alpha\beta} - \sum_{k=1}^{m} x_{k\alpha}x_{k\beta}\right). \tag{4.3.8}$$

We are going to evaluate the Beltrami operator

$$\Delta = \mathfrak{G}^{(i\alpha)(j\beta)}\left(\frac{\partial^2}{\partial x_{i\alpha}\partial x_{j\beta}} - \left\{ \begin{matrix} (k\gamma) \\ (i\alpha)(j\beta) \end{matrix} \right\} \frac{\partial}{\partial x_{k\gamma}}\right). \tag{4.3.9}$$

Here we use the summation convention. The Latin letters $i, j, \cdots$ run from 1 to m and the Greek letters $\alpha, \beta, \cdots$ run from 1 to n.

For simplicity we denote

$$\begin{aligned} &I - XX' = (a_{ij}), \quad (I - XX')^{-1} = (A_{ij}), \\ &I - XX' = (b_{\alpha\beta}), \quad (I - X'X)^{-1} = (B_{\alpha\beta}). \end{aligned} \tag{4.3.10}$$

Since

$$\frac{\partial g_{(i\alpha)(j\beta)}}{\partial x_{l\lambda}} = - g_{(i\alpha)(p\mu)}g_{(j\beta)(q\nu)}\frac{\partial g^{(p\mu)(q\nu)}}{\partial x_{l\lambda}}$$
$$=g_{(i\alpha)(p\mu)}g_{(j\beta)(q\nu)}[(\delta_{lp}x_{q\lambda}+\delta_{lq}x_{p\lambda})(\delta_{\nu\mu}-x_{s\nu}x_{s\mu})$$
$$+(\delta_{pq}-x_{p\sigma}x_{q\sigma})(\delta_{\lambda\mu}x_{l\nu}+\delta_{\lambda\nu}x_{l\mu})],$$

we have

$$g^{(i\alpha)(j\beta)}g^{(k\gamma)(l\lambda)}\frac{\partial g_{(i\alpha)(j\beta)}}{\partial x_{l\lambda}}$$
$$=g^{(k\gamma)(l\lambda)}g_{(p\mu)(q\nu)}[(\delta_{lp}x_{q\lambda}+\delta_{lq}x_{p\lambda})b_{\mu\nu}+a_{pq}(\delta_{\lambda\mu}x_{l\nu}+\delta_{\lambda\nu x_{l\mu}})]$$
$$=a_{kl}b_{\gamma\lambda}A_{pq}B_{\mu\nu}[(\delta_{lp}x_{q\lambda}+\delta_{lq}x_{p\lambda})b_{\mu\nu}+a_{pq}(\delta_{\lambda\mu}x_{l\nu}+\delta_{\lambda\nu x_{l\mu}})]$$
$$=n(a_{kl}b_{\gamma\lambda}A_{pq}\delta_{lp}x_{q\lambda}+a_{k\lambda}b_{\gamma\lambda}A_{pq}\delta_{lq}x_{p\lambda})$$
$$+m(a_{kl}b_{\gamma\lambda}B_{\mu\nu}\delta_{\lambda\mu}x_{l\nu}+a_{kl}b_{\gamma\lambda}B_{\mu\nu}\delta_{\lambda\nu}x_{l\mu})$$
$$=2nb_{\gamma\lambda}x_{k\lambda}+2ma_{kl}x_{l\gamma}. \tag{4.3.11}$$

Similarly,

$$2g^{(i\alpha)(j\beta)}g^{(k\gamma)(l\lambda)}\frac{\partial g_{(i\alpha)(l\lambda)}}{\partial x_{j\beta}}$$
$$=2g^{(i\alpha)(j\beta)}g^{(k\gamma)(l\lambda)}g_{(i\alpha)(p\mu)}g_{(l\lambda)(q\nu)}[(\delta_{jp}x_{q\beta}+\delta_{jq}x_{p\beta})b_{\mu\nu}$$
$$+a_{pq}(\delta_{\beta\mu}x_{j\nu}+\delta_{\beta\nu}x_{j\mu})]$$
$$=2(m+1)x_{k\mu}b_{\mu\gamma}+2(n+1)a_{pk}x_{p\gamma}. \tag{4.3.12}$$

Substituting (4.3.11) and (4.3.12) into (4.3.9), we obtain

$$g^{(i\alpha)(j\beta)}\begin{Bmatrix}(k\gamma)\\(i\alpha)(j\beta)\end{Bmatrix}=(m+1)x_{k\mu}b_{\mu\gamma}+(n+1)a_{kp}x_{p\gamma}-nb_{\gamma\lambda}x_{k\lambda}-ma_{kl}x_{l\gamma}$$
$$=(n-m+1)a_{kl}x_{p\gamma}-(n-m-1)x_{k\mu}b_{\mu\gamma}$$
$$=(n-m+1)(\delta_{kp}-x_{k\sigma}x_{p\sigma})x_{p\gamma}-(n-m-1)x_{k\mu}(\delta_{\mu\gamma}-x_{l\mu}x_{l\gamma})$$
$$=2(\delta_{kp}-x_{k\sigma}x_{p\sigma})x_{p\gamma}.$$

Hence the Beltrami operator of $\mathfrak{R}$ becomes

$$\Delta=\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n}\left(\delta_{ij}-\sum_{\gamma=1}^{n}x_{i\gamma}x_{j\gamma}\right)\left(\delta_{\alpha\beta}-\sum_{k=1}^{m}x_{k\alpha}x_{k\beta}\right)\frac{\partial^2}{\partial x_{i\alpha}\partial x_{j\beta}}$$
$$-2\sum_{k,p=1}^{m}\sum_{\gamma=1}^{n}x_{p\gamma}\left(\delta_{kp}-\sum_{\sigma=1}^{n}x_{k\sigma}x_{p\sigma}\right)\frac{\partial}{\partial x_{k\gamma}}. \tag{4.3.9$'$}$$

If we introduce the matrix operator

$$\partial_X = \begin{pmatrix} \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{12}} & \cdots & \frac{\partial}{\partial x_{1n}} \\ \frac{\partial}{\partial x_{21}} & \frac{\partial}{\partial x_{22}} & \cdots & \frac{\partial}{\partial x_{2n}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial x_{m1}} & \frac{\partial}{\partial x_{m2}} & \cdots & \frac{\partial}{\partial x_{mn}} \end{pmatrix} \tag{4.3.13}$$

then

$$\Delta = \mathrm{tr}[(I - XX')\partial_X(I - X'X)\partial_X' - 2(I - XX')X\partial_X']. \qquad (4.3.9)''$$

Since Δ is invariant under $\mathfrak{G}$, obviously we have

Theorem 4.3.1 *Any harmonic function in $\mathfrak{R}$ remains harmonic after the transformation of $\mathfrak{G}$.*

Let $\mathfrak{B}^{(0)}$ be the closure of $\mathfrak{R}$ and $\mathfrak{B}^{(m-r)}$ be the set of matrices X such that $I - XX'$ is positive semi-definite and of rank $\leqslant r$. Then we have

Theorem 4.3.2 *$\mathfrak{B}^{(0)} \supset \mathfrak{B}^{(1)} \supset \cdots \supset \mathfrak{B}^{(m-1)}$ form a chain of slit spaces. The slit of $\mathfrak{B}^{(m-1)}$ is $\mathfrak{C} = \mathfrak{C}(m, n)$ which is the set of real $m \times n$ matrices Γ such that*

$$\Gamma\Gamma' = I^{(m)}. \tag{4.3.14}$$

More precisely, the set $\mathfrak{C}^{(r)} = \mathfrak{B}^{(m-r)} - \mathfrak{B}^{(m-r+1)}$ is homeomorphic to the topological product

$$\mathfrak{R}(r, n - m + r) \times \mathfrak{M}^{(m-r)},$$

where $\mathfrak{M}^{(m-r)}$ is defined in the following way: A pair of matrices $(\Gamma_1, \Gamma_2), \Gamma_1 \in O(m), \Gamma_2 \in O(n)$, is said to be equivalent to the pair $(\overset{*}{\Gamma}_1, \overset{*}{\Gamma}_2), \overset{*}{\Gamma}_1 \in O(m), \overset{*}{\Gamma}_2 \in O(n)$, if

$$\Gamma_1 = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \overset{*}{\Gamma}_1, \quad \Gamma_2 = \begin{pmatrix} A & O \\ O & C \end{pmatrix} \overset{*}{\Gamma}_2,$$

where A, B, C are $(m - r) \times (m - r), r \times r, (n - m + r) \times (n - m + r)$ real orthogonal matrices respectively. We identify the equivalent pairs and form a quotient space which we denote by $\mathfrak{M}^{(m-r)}$.

Proof It should be noticed that the matrix X of $\mathfrak{C}^{(r)}$ can be written as

$$X = \Gamma_1' \begin{pmatrix} I^{(m-r)} & O \\ O & X_1 \end{pmatrix} \Gamma_2,$$

where $\Gamma_1 \in O(m), \Gamma_2 \in O(n)$ and $I^{(r)} - X_1X_1' > O$. The remaining proof of this theorem is analogous to that of Theorem 2.3.3.

Similar to Theorem 2.3.1, we can prove

Theorem 4.3.3 *$\mathfrak{C}^{(r)}$ is invariant under $\mathfrak{G}$ and any point of $\mathfrak{C}^{(r)}$ can be transformed by $\mathfrak{G}$ into the point*

$$\begin{pmatrix} I^{(m-r)} & O \\ O & O^{(r,n-m+r)} \end{pmatrix}$$

A function $u(X)$ is said to be harmonic on the closure of $\mathfrak{R}$, if it is continuous in $\mathfrak{B}^{(0)}$ and on each $\mathfrak{C}^{(r)}(r = 1, \cdots, m)$ it is harmonic with respect to the coordinate of $\mathfrak{R}(r, n-m+r)$.

A corresponding Dirichlet problem is solved:

Theorem 4.3.4 *If a continuous function $\varphi(\Gamma)$ is given in $\mathfrak{C}$, then the function*

$$u(X) = \frac{1}{V(\mathfrak{C}(m,n))} \int_{\mathfrak{C}(m,n)} \varphi(\Gamma) \frac{\det(I - XX')^{\frac{n-1}{2}}}{\det(I - X\Gamma')^{n-1}} \dot{\Gamma},$$

$$V(\mathfrak{C}(m,n)) = \prod_{\nu=n-m+1}^{n} \omega_{\nu-1} \tag{4.3.15}$$

is the unique function harmonic on the closure of $\mathfrak{R}$, which takes the given boundary value $\varphi(\Gamma)$ on $\mathfrak{C}$.

Proof (i) At first we prove that $u(X)$ is harmonic in $\mathfrak{R}(m, n)$. It is sufficient to prove that

$$\Delta \circ \frac{\det(I - XX')^{\frac{n-1}{2}}}{\det(I - X\Gamma')^{n-1}} = 0$$

with respect to X.

Since after the transformation (4.3.2) we have

$$\frac{\det(I - YY')^{\frac{n-1}{2}}}{\det(I - Y\Gamma_1')^{n-1}} = \frac{\det(I - XX')^{\frac{n-1}{2}}}{\det(I - X\Gamma')^{n-1}} \det(B\Gamma' + A)^{n-1},$$

where

$$\Gamma_1 = (A\Gamma_1 + B)(C\Gamma_1 + D)^{-1},$$

it should remain only to prove

$$\left[\Delta \circ \frac{\det(I - XX')^{\frac{n-1}{2}}}{\det(I - X\Gamma')^{n-1}}\right]_{X=0} = 0.$$

In fact, suppose that $\Gamma = (\gamma_{i\alpha})$, then

$$\left[\Delta \circ \frac{\det(I - XX')^{\frac{n-1}{2}}}{\det(I - X\Gamma')^{n-1}}\right]_{X=0} = \left[\sum_{i=1}^{m}\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x_{i\alpha}\partial x_{i\alpha}} \circ \frac{\det(I - XX')^{\frac{n-1}{2}}}{\det(I - X\Gamma')^{n-1}}\right]_{X=0}$$

$$=\left[\sum_{i=1}^{m}\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x_{i\alpha}^2}\left\{1 - \frac{n-1}{2}\sum_{i=1}^{m}\sum_{\alpha=1}^{n} x_{i\alpha}^2 + \cdots\right\}\right.$$

$$\times\left\{1 + (n-1)\sum_{i=1}^{m}\sum_{\alpha=1}^{n} x_{i\alpha}\gamma_{i\alpha} - \frac{n-1}{2}\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n}(x_{i\alpha}\gamma_{i\alpha}x_{i\beta}\gamma_{j\beta} - x_{i\alpha}\gamma_{j\alpha}x_{j\beta}\gamma_{i\beta})\right.$$

$$\left.\left. + \frac{(n-1)n}{2}\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} x_{i\alpha}\gamma_{i\alpha}x_{j\beta}\gamma_{j\beta} + \cdots\right\}\right]_{X=0}$$

$$= -\frac{n-1}{2}\sum_{i=1}^{m}\sum_{\alpha=1}^{n} 2 - \frac{n-1}{2}\sum_{i=1}^{m}\sum_{\alpha=1}^{n}(\gamma_{i\alpha}\gamma_{i\alpha} - \gamma_{i\alpha}\gamma_{i\alpha})$$

$$+ \frac{n(n-1)}{2}\sum_{i=1}^{m}\sum_{\alpha=1}^{n} 2\gamma_{i\alpha}\gamma_{i\alpha}$$

$$= -(n-1)mn + n(n-1)m = 0.$$

Hence $u(X)$ is harmonic in $\mathfrak{R}$.

(ii) We want to prove that $u(X)$ in $\mathfrak{C}^{(r)}$ is harmonic with respect to the coordinate of $\mathfrak{R}(r, n-m+r)$; i.e., if $Q = \Gamma_1' \begin{pmatrix} I^{(m-r)} & 0 \\ 0 & X_0 \end{pmatrix} \Gamma_2$,

$$\lim_{X \to Q} u(X) = \frac{1}{V(\mathfrak{C}(r, n-m+r))}\int_{\mathfrak{C}(r,n-m+r)} \varphi\left(\Gamma_1'\begin{pmatrix} I^{(m-r)} & 0 \\ 0 & \Gamma_0 \end{pmatrix}\Gamma_2\right)$$
$$\times \frac{\det(I - X_0X_0')^{\frac{n-m+r-1}{2}}}{\det(I - X_0\Gamma_0')^{n-m+r-1}}\dot{\Gamma}_0. \tag{4.3.16}$$

It is sufficient to prove the particular case that for $r = m-1, X_0 = 0, \Gamma_1 = I^{(m)}, \Gamma_2 = I^{(n)}$, and $X = \begin{pmatrix} \rho & 0 \\ 0 & O \end{pmatrix}, 0 \leqslant \rho < 1$,

$$\lim_{\rho \to 1} \frac{1}{V(\mathfrak{C}(m,n))}\int_{\mathfrak{C}(m,n)} \varphi(\Gamma)\frac{(1-\rho^2)^{\frac{n-1}{2}}}{(1-\rho\gamma_{11})^{n-1}}\dot{\Gamma}$$
$$= \frac{1}{V(\mathfrak{C}(m-1,n-1))}\int_{\mathfrak{C}(m-1,n-1)} \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & \Gamma_0 \end{pmatrix}\right)\dot{\Gamma}_0.$$

Denote $\Gamma = \begin{pmatrix} \gamma \\ \Gamma_1 \end{pmatrix}$, where $\gamma = (\gamma_{11}, \cdots, \gamma_{1n})$ satisfies $\gamma\gamma' = 1$.

Let

$$\psi(\gamma) = \int_{\begin{pmatrix}\gamma\\ \Gamma_1\end{pmatrix}\begin{pmatrix}\gamma\\ \Gamma_1\end{pmatrix}'=I} \varphi\left(\begin{pmatrix}\gamma\\ \Gamma_1\end{pmatrix}\right)\dot{\Gamma}_1.$$

According to a theorem analogous to Theorem 1.3.2, we can prove

$$\begin{aligned}
&\lim_{\rho\to 1}\frac{1}{V(\mathfrak{C}(m,n))}\int_{\mathfrak{C}(m,n)}\varphi(\Gamma)\frac{(1-\rho^2)^{\frac{n-1}{2}}}{(1-\rho\gamma_{11})^{n-1}}\dot{\Gamma}\\
=&\lim_{\rho\to 1}\frac{1}{V(\mathfrak{C}(m,n))}\int_{\gamma\gamma'=1}\psi(\gamma)\frac{(1-\rho^2)^{\frac{n-1}{2}}}{(1-\rho\gamma_{11})^{n-1}}\dot{\gamma}\\
=&\frac{\omega_{n-1}}{V(\mathfrak{C}(m,n))}\int_{\mathfrak{C}(m-1,n-1)}\varphi\left(\begin{pmatrix}1 & 0\\ 0 & \Gamma_0\end{pmatrix}\right)\dot{\Gamma}_0\\
=&\frac{1}{V(\mathfrak{C}(m-1,n-1))}\int_{\mathfrak{C}(m-1,n-1)}\varphi\left(\begin{pmatrix}1 & 0\\ 0 & \Gamma_0\end{pmatrix}\right)\dot{\Gamma}_0.
\end{aligned}$$

Moreover, according to formula (4.3.16), we know that $u(X)$ takes the given boundary value $\varphi(\Gamma)$. Again, by Theorem 1.2.2, $u(X)$ is the unique solution satisfying the conditions of our theorem.

References

Bergmann, S. [1] Kernel function and extended class in the theory of functions of complex variables. *Colloque sur les fonc. plu. variables,* 1953: 135–157.

Duff, G. F. D. and Spencer, D. C. [1] *Ann. of Math.*, 1952, **56**: 128–168.

Hodge, W. V. D. [1] 1952 *The theory and applications of harmonic integrals.* 2nd ed.

Hua, L. K.

[1] *Amer. J. Math.*, 1944, **66**: 470–488.

[2] *Ann. of Math.*, 1946, **47**: 167–191.

[3] *Science Record,* 1957, *New Ser.*, **1**: 7–9.

[4] *Harmonic analysis of the classical domains of analytic functions of several complex variables.* Peking (in Chinese), 1958.

[5] *Science Record,* New Ser., 1958, **2**: 281–283.

Hua, L. K. and Look, K. H.

[1] *Science Record,* New Ser., 1958, **2**: 19–22.

[2] *Science Record,* New Ser., 1958, **2**: 77–80.

[3] *Acta Math. Sinica,* 1958, **8**: 531–547 (in Chinese).

[4] *Acta Math. Sinica,* 1958, **9** (in Chinese).

[5] *Acta Math. Sinica,* 1958, **9** (in Chinese).

[6] Theory of harmonic functions in classical domains IV. Harmonic functions in the hyperbolic space of Lie-sphere (to appear).

Look, K. H.

[1] *Unitary geometry in the theory of functions of several complex variables*, 1956 (in Chinese).

[2] *Science Record,* New Ser., 1959, **3**: 289–294.

Lowdenslager, D. B. [1] 1958, *Ann. of Math.*, **67**: 467–484.

Miranda, C. [1] *Equazion alle derivate parziali di tipo ellittico*, 1955.

Mitchell, J. [1] *Trans. Amer. Soc.*, 1955, **79**: 401–422.

ON FOURIER TRANSFORMS IN L^p IN THE COMPLEX DOMAIN*

BY LOO-KENG HUA AND SHIEN-SIU SHÜ

Introduction.

In this paper we are concerned with functions belonging to H^pL^P as introduced by A. C. Offord①. In particular, when $p = 2$, the class H^2L^2 is identical with L^2. R. E. A. C. Paley and N. Wiener② have obtained some very beautiful results in the case of L^2 on the Fourier transform of a function vanishing exponentially, of a function analytic in a strip, of a function analytic in a half-plane, etc. The purpose of this paper is to extend those results to H^pL^p. A difficulty in making the generalization is that for H^pL^p we have no corresponding Plancherel theorem

$$\int_{-\infty}^{\infty} |F(x)|^2 dx = \int_{-\infty}^{\infty} |f(u)|^2 du,$$

where $F(x)$ is the Fourier transform of $f(u)$. The main results may be summarized as following:

1) If $f(u)$ is a function belonging to L^p in any finite interval and

$$f(x) = \begin{cases} O(e^{-\mu u} K(u)) & (u \to \infty), \\ O(e^{\lambda u} K(u)) & (u \to -\infty), \end{cases}$$

where $K(u)$ is a constant or a function of u belonging to $L^p(-\infty, \infty)$ then $f(u)$ belongs to $H^pL^p, p \geqslant 2$(Theorems 1 and 2).

2) If $F(s)$ is analytic in a strip, if $F(s)$ belongs uniformly to L^p in the strip and if $F(s)$ belongs to H^p on the boundaries of the strip, then for any interior point s, $F(s)$ can be expressed by the associated Cauchy integrals, and $F(s)$ belongs to H^pL^p in the strip (Theorems 3 and 5).

3) The conclusions of 2) are valid if we replace the condition that $F(s)$ belongs

* Reprinted from the *Journal of Mathematics and Physics*, 1936, 15: 249–263.

By Loo-Keng Hua and Shien-siu Shu. National Tsing Hua University, Peiping, China.

① A. C. Offord. On Fourier Transforms III. *Trans. Amer. Math. Soc.*, 1935, 38: 250–266.

② Paley and Wiener. Fourier Transforms in the Complex Domain. *Amer. Math. Soc. Colloquium Publication*, vol 19.

uniformly to L^p in the strip by the requirement that

$$F(s) = O\left(e^{e\rho|t|}\right)$$

(Theorem of Phragmén-Lindelöf type; Theorem 7).

4) If $F(s)$ is analytic over the right half-plane if $F(s)$ belongs uniformly to L^p or

$$\lim_{r\to\infty}\frac{1}{r}\log|F(re^{\theta i})| = 0$$

uniformly over the right half-plane and if $F(it)$ belongs to H^p, then for any interior s, $F(s)$ can be expressed by an associated Cauchy integral and $F(s)$ belongs to H^pL^p (Theorems 8 and 10).

5) The two following classes of entire functions are identical:

(a) the class of entire functions $F(s)$ belonging to H^pL^p along the real axis and satisfying the condition

$$F(s) = O\left(e^{A|s|}\right),$$

(b) the class of all entire functions of the form $F(s) = \int_{-A}^{A} f(u)e^{ius}du$, where $f(u)$ belongs to L^p over $(-A, A)$ (Theorem 11).

6) If $F(z)$ is an entire function such that

$$\lim_{r\to\infty}\frac{1}{r}\log^+|F(re^{\theta i})| = 0$$

and does not vanish identically, it can not belong to H^pL^p along any line (Theorem 12).

We are indebted to Prof. N. Wiener for his lectures on Fourier Transforms at Tsing Hua University and we must take this opportunity to express our deep gratitude for his many valuable suggestions.

§1. In this section let us assume $f(u)$ to be a function belonging to L^p in any finite interval, $p \geqslant 2$, and

$$f(u) = \begin{cases} O(e^{-\mu u}K(u)) & (u\to\infty), \\ O(e^{\lambda u}K(u)) & (u\to-\infty), \end{cases} \tag{1.1}$$

where $K(u)$ is a constant or a function belonging to $L^p(-\infty,\infty)$.

THEOREM 1 Let $\delta > 0$ and $-\lambda+\delta \leqslant \sigma \leqslant \mu-\delta$. Then

$$F(\sigma, t) = \int_{-\infty}^{\infty} f(u)e^{\sigma u - itu}du \tag{1.2}$$

exists and is bounded uniformly over the strip, and

$$F(\sigma, -t) = F(z), \quad z = \sigma + it,$$

is an analytic function of z.

Proof It is not very difficult to prove that $f(u)e^{\sigma u}$ belongs to L^{p_1} where $0 < p_1 \leqslant p$. In particular, $p_1 = 1$, we see that (1.2) exists. Furthermore, over $-\lambda + \delta \leqslant \sigma \leqslant \mu - \delta$, we have

$$\begin{aligned}|F(\sigma,t)| &\leqslant \int_{-\infty}^{\infty} |f(u)|e^{\sigma u}du \\ &\leqslant \text{const.} \left[\int_{-\infty}^{-N} |K(u)|^{e_{\delta u}}du + \int_{N}^{\infty} |K(u)|e^{-\delta u}du\right] \\ &\quad + \int_{-N}^{0} |f(u)|e^{-\lambda u}du + \int_{0}^{N} |f(u)|e^{\mu u}du \\ &= \text{const.(Independent of } \sigma \text{ and } t).\end{aligned}$$

Next let us consider the function

$$\begin{aligned}&\int_{-\infty}^{\infty} f(u)\left(\frac{e^{(z+\Delta z)u} - e^{zu}}{\Delta z} - ue^{zu}\right)du \\ =&\int_{-\infty}^{\infty} uf(u)e^{zu}\left(\frac{e^{\Delta zu}-1}{\Delta zu} - 1\right)du \\ =&\int_{-\infty}^{0} uf(u)e^{(z-\frac{\delta}{2})u}e^{\frac{\delta}{2}u}\left(\frac{e^{\Delta zu}-1}{\Delta zu} - 1\right)du \\ &+\int_{0}^{\infty} uf(u)e^{(z+\frac{\delta}{2})u}e^{-\frac{\delta}{2}u}\left(\frac{e^{\Delta zu}-1}{\Delta zu} - 1\right)du.\end{aligned}$$

We are now going to prove that when $u < 0, |R(\Delta z)| < \frac{\delta}{2}$, we have

$$|G(u,\Delta z)| = \left|e^{\frac{\delta}{2}u}\left(\frac{e^{\Delta zu}-1}{\Delta zu} - 1\right)\right| \leqslant 3.$$

In fact, when $|\Delta zu| \geqslant 1$,

$$|G(u,\Delta z)| \leqslant e^{R(\frac{\delta}{2}+\Delta z)u} + 2e^{\frac{\delta}{2}u} \leqslant 3$$

when $|\Delta zu| < 1$,

$$|G(u,\Delta z)| \leqslant \left|\frac{u\Delta z}{2!} + \frac{(u\Delta z)^2}{3!} + \cdots\right|$$

$$\leqslant \frac{1}{2!} + \frac{1}{3!} + \cdots \leqslant e - 1 < 3.$$

Similarly when $u > 0, |R(\Delta z)| < \frac{\delta}{2}$, we have

$$\left| e^{-\frac{\delta}{2}u} \left(\frac{e^{\Delta z u} - 1}{\Delta z u} - 1 \right) \right| \leqslant 3.$$

Using Lebesgue's convergence theorem we have that (1.3) tends to zero as $\Delta z \to 0$.Thus

$$\frac{d}{dz} \int_{-\infty}^{\infty} f(u) e^{zu} du = \int_{-\infty}^{\infty} f(u) u e^{zu} du.$$

THEOREM 2 *Under the same hypotheses as before, we have*

$$\int_{-\infty}^{\infty} |F(\sigma, t)|^p dt < C,$$

*where C is a constant independent of σ. Thus, by Offord's criterion of $H^p, f(u)e^{\sigma u}$ belongs to $H^p L^p$*③*.Consequently $F(\sigma, t)$ belongs to $H^p L^p$*④.

Proof Firstly, let us prove that there exists a number T such that $|t| > T$,

$$|F(\sigma, t)| < 1$$

for any σ on $-\lambda + \delta \leqslant \sigma \leqslant \mu - \delta$. Suppose this be false when $t \to \infty$. Let $t_1 < t_2 < \cdots < t_n \cdots$ be a sequence which tends to infinity. Then for each t_i, there is at least one $\sigma_i(-\lambda + \delta \leqslant \delta_i \leqslant \mu - \delta)$ and one $\tau_i(> t_i)$ such that

$$|F(\sigma_i, \tau_i)| \geqslant 1.$$

Let σ_0 be one of the limiting points of $\{\sigma_i\}$ (It does exist by the Bolzano-Weierstrass theorem).

By the Riemann-Lebesgue Theorem there exists a number t_0, such that when $t > t_0$, we have

$$|F(\sigma_0, t)| < \frac{1}{4}.$$

On the other hand, since $F(\sigma, -t)$ is an analytic function of $\sigma + it$

$$|F(\sigma + \Delta\sigma, t) - F(\sigma, t)| \leqslant \left| \int_{\sigma}^{\sigma+\Delta\sigma} F'_{\sigma}(\sigma, t) dt \right|$$

③ Loc. cit Th. 9.
④ Loc. cit Th. 12.

$$\leqslant \int_{\sigma}^{\sigma+\Delta\sigma} \left| \int_{-\infty}^{\infty} u f(u) e^{(\sigma-it)u} du \right| dt.$$

The integrand is uniformly bounded over the strip $-\lambda+\delta \leqslant \sigma \leqslant \mu-\delta$. Hence we can choose δ_0, such that whenever

$$|\sigma - \sigma_0| < \delta_0,$$

we obtain

$$|F(\sigma, t) - F(\sigma_0, t)| < \frac{1}{4}$$

for any value of t. Therefore

$$|F(\sigma, t)| < \frac{1}{2} \quad (t > t_0).$$

We can then choose m great enough, such that $|\sigma_m - \sigma_0| < \sigma_0$ and $t_m > t_0$, thus we have

$$F(\sigma_m, \tau_m)| < \frac{1}{2}.$$

This is a contradiction.

Now let us consider

$$\int_{-\infty}^{\infty} |F(\sigma, t)|^p dt = \left[\int_{-\infty}^{-T} + \int_{-T}^{T} + \int_{T}^{\infty} \right] |F(\sigma, t)|^p dt$$
$$\leqslant \int_{-T}^{T} |F(\sigma, t)|^p dt + \left[\int_{-\infty}^{-T} + \int_{T}^{\infty} \right] |F(\sigma, t)|^2 dt.$$

The first term is easily seen to be bounded, since its integrand is bounded by theorem 1. The second term is uniformly bounded over the given strip by the Plancherel theorem in L^2.

§2. **Theorem 3** *If $F(s)$ is analytic over $-\lambda \leqslant \sigma \leqslant \mu, s = \sigma + it$ and*

$$\int_{-\infty}^{\infty} |F(\sigma + it)|^p dt < \text{const.}$$

over this region, then when s is an interior point of this region, we have

$$F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(\mu + iy)}{\mu + iy - s} dy - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(-\lambda + iy)}{-\lambda + iy - s} dy. \tag{2.1}$$

The proof of this theorem is quite similar to that given in N. Wiener and Paley's book (pp.3-5) this could be done merely by using Hölder inequality instead of Schwarz inequality.

Consequently we have

THEOREM 4 *Under the hypotheses of Theorem 3, $F(s)$ is bounded over any region $-\lambda+\delta \leqslant \sigma \leqslant \mu-\delta$.*

By the Hölder inequality we obtain that

$$\left|\int_{-\infty}^{\infty} \frac{F(\mu+iy)}{\mu+iy-s} dy\right| \leqslant \left\{\int_{-\infty}^{\infty} |F(\mu+iy)|^p dy\right\}^{\frac{1}{p}} \left\{\int_{-\infty}^{\infty} \frac{dy}{|\mu+iy-s|^q}\right\}^{\frac{1}{q}}$$

is bounded since $2<p<\infty$ and $q=\dfrac{p}{p-1}>1$. Similarly, the other term of (2.1) is also bounded.

THEOREM 5 *Besides the hypotheses of Theorem 3, we assume further that $F(\sigma+it)$ belongs to H^p for $\sigma=\mu$ and $-\lambda$, then there exists a measurable function $f(x)$ such that*

$$\int_{-\infty}^{\infty} |f(x)|^p e^{p\mu x} dx < M; \quad \int_{-\infty}^{\infty} |f(x)|^p e^{-p\lambda x} dx < M, \tag{2.2}$$

and that over the open interval $-\lambda<\sigma<\mu, F(\sigma+it)$ and $f(x)e^{\sigma x}$ are Fourier transforms of each other in L^p. Moreover, for each σ on this open interval

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{(\sigma+it)u} du$$

converges in the ordinary sense to the function $F(\sigma+it)$ everywhere on $-\infty<t<\infty$.

Proof Since $F(\sigma+it)$ belongs to H^pL^p for $\sigma=\mu$ and $\sigma=-\lambda$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\mu+it) e^{-itx} dt$$

is summable $(C,1)$ almost everywhere to a function $f(\mu,x)$ which is the Fourier transform of $F(\mu+it)$ in L^p and therefore it belongs also to H^pL^p. Similarly we define $f(-\lambda,t)$. (Offord, Theorem 12).

Let us put

$$G(x)=\begin{cases} 0, & x<0; \\ e^{-\alpha x+itx}, & x>0; \alpha>0. \end{cases}$$

We obtain

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x) e^{ixy} dx &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ixy-(\alpha-it)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha-it-iy} = g(y),\end{aligned}$$

$g(y)$ is bounded in any finite range. By Theorems 1 and 2, $G(x)$ belongs to $H^qL^q, q = \frac{p}{p-1}$. By Offord's Theorem 1,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-ixy}}{\alpha - i(t+y)}dy = \begin{cases} 0, & x<0, \\ e^{(-\alpha+it)x}, & x>0. \end{cases} \tag{c, 1}$$

Again by the use of Offord's Theorem 3, when $-\lambda+\delta \leqslant \sigma \leqslant \mu-\delta$, and for any value of t we have

$$\int_0^{\infty} f(\mu,x)e^{(\sigma-\mu)x}e^{itx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(\mu+iy)}{\mu+iy-(\sigma+it)}dy$$

by putting $\alpha = \mu - \sigma$. It is easily seen that the two integrals of both sides converge in the ordinary sense. Similarly,

$$\int_{-\infty}^{0} f(-\lambda,x)e^{(\sigma+\lambda)x}e^{itx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{F(-\lambda+iy)}{-\lambda+iy-(\sigma+it)}dy.$$

By Theorem 3, we get

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(-\lambda,x)e^{(\sigma+\lambda)x}e^{itx}dx + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(\mu,x)e^{(\sigma-\mu)x}e^{itx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{\sigma x}dx, \end{aligned} \tag{2.3}$$

where

$$f(x) = \begin{cases} f(-\lambda,x)e^{\lambda x} & (x<0), \\ f(\mu,x)e^{-\mu x} & (x>0) \end{cases} \tag{2.4}$$

and (2.3) converges everywhere in t in the ordinary sense. By Theorem 1, for $-\lambda < \sigma < \mu, f(x)e^{\sigma x}$ and $F(s)$ belong to L^pH^p, Moreover

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^p e^{p\mu x}dx &= \int_{-\infty}^{0} |f(-\lambda,x)|^p e^{p(\mu+\lambda)x}dx + \int_0^{\infty}|f(\mu,x)|^p dx \\ &\leqslant \int_{-\infty}^{0}|f(-\lambda,x)|^p dx + \int_0^{\infty}|f(\mu,x)|^p dx \\ &< \text{const}. \end{aligned}$$

Similarly

$$\int_{-\infty}^{\infty}|f(x)|^p e^{-p\lambda x}dx < \text{const}.$$

§3. Theorems of the Phragmén-Lindelöf type

Theorem 6 *If $F(\sigma + it)$ be analytic over $-\lambda \leqslant \sigma \leqslant \mu$, and $F(\sigma + it)$ belongs to $H^p L^p$ for $\sigma = -\lambda$ and $\sigma = \mu, p \geqslant 2$, and if*

$$|F(\sigma + it)| < M$$

uniformly in the strip $-\lambda \leqslant \sigma \leqslant \mu$, then the conclusions of Theorems 3 and 5 are valid.

Proof The conclusion of Theorem 3 is easily verified, just as shown in Paley and Wiener's book, p. 9.

In the proof of Theorem 5, we have only used the facts that the conclusions of Theorem 3 is valid and that $F(\sigma + it)$ belongs to $H^p L^p$ for $\sigma = -\lambda$ and $\sigma = \mu$. These are true in the present case.

Theorem 7 If $F(s)$ is an analytic function of s over the strip $-\lambda \leqslant \sigma \leqslant \mu$, if $F(s)$ belongs to $H^p L^p (p \geqslant 2)$ for $\sigma = -\lambda$ and $\sigma = \mu$, and if

$$|F(\sigma + it)| = O\left(e^{e^{\rho|t|}}\right) \quad (-\lambda \leqslant \sigma \leqslant \mu),$$

where $\rho < \pi/(\lambda + \mu)$, then the conclusions of Theorems 3, 4 and 5 are valid.

Proof Let us consider the function

$$F_\varepsilon(\sigma + it) = \frac{1}{\epsilon}\int_t^{t+\varepsilon} F(\sigma + it)dt,$$

which satisfies the following four properties:

$$F_\varepsilon(\sigma + it) = O\left(e^{e^{\rho'|t|}}\right) \quad (-\lambda < \sigma < \mu) \tag{1}$$

$$|F_\varepsilon(\mu + it)| \leqslant \frac{1}{\varepsilon^{1-\frac{1}{q}}}\left[\int_t^{t+\varepsilon} |F(\mu + it)|^p dt\right]^{\frac{1}{p}},$$

$$|F_\varepsilon(-\lambda + it)| \leqslant \frac{1}{\varepsilon^{1-\frac{1}{q}}}\left[\int_t^{t+\varepsilon} |F(-\lambda + it)|^p dt\right]^{\frac{1}{p}}, \quad q = \frac{p-1}{p}. \tag{2}$$

Thus, for each $\varepsilon > 0, F_\varepsilon(\sigma + it)$ is uniformly bounded over $-\lambda \leqslant \sigma \leqslant \mu$ by the classical Phragmén-Lindelöf theorem.

(3) $F_\varepsilon(\mu + it)$ and $F_\epsilon(-\lambda + it)$ belong to L^p uniformly in ϵ. Since by the Hölder inequality, we obtain

$$\int_a^b \left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} F(\mu + iy)dy\right|^p dt \leqslant \frac{1}{\varepsilon^{p-\frac{p}{q}}}\int_a^b dt \int_t^{t+\varepsilon} |F(\mu + iy)|^p dy$$

$$=\frac{1}{\varepsilon}\int_{a+\varepsilon}^{b}dy\int_{y-\varepsilon}^{y}|F(\mu+iy)|^p dt+\frac{1}{\varepsilon}\int_{a}^{a+\varepsilon}dy\int_{a}^{y}|F(\mu+iy)|^p dt$$
$$+\frac{1}{\varepsilon}\int_{b}^{b+\varepsilon}dy\int_{y-\varepsilon}^{b}|F(\mu+iy)|^p dt$$
$$<\int_{a}^{b+\varepsilon}|F(\mu+iy)|^p dy$$
$$\leqslant\int_{-\infty}^{\infty}|F(\mu+iy)|^p dy.$$

Similarly, $F_\varepsilon(-\lambda+it)$ belongs to L^p uniformly in ε.

(4) For each $\varepsilon>0, F_\varepsilon(\mu+it)$ and $F_\varepsilon(-\lambda+it)$ belong to H^p. Let us put

$$F(\mu+it)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\mu,x)e^{ixt}dx, \qquad (C,1)$$

where $f(\mu,x)$ belongs to H^pL^p. Therefore

$$F(\mu+it)-\frac{1}{\sqrt{2\pi}}\int_{-w}^{w}\left(1-\frac{|x|}{w}\right)f(\mu,x)e^{itx}dx=o(1)$$

as $w\to\infty$. Thus we have

$$F_\varepsilon(\mu+it)-\frac{1}{\sqrt{2\pi}}\int_{-w}^{w}\left(1-\frac{|x|}{w}\right)f(\mu,x)\frac{e^{i\varepsilon x}-1}{i\varepsilon x}e^{ixt}dx=o(1)$$

almost everywhere in t. Hence

$$F_\varepsilon(\mu+it)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\mu,x)\frac{e^{i\varepsilon x}-1}{i\varepsilon x}e^{ixt}dx.$$

The right integral converges everywhere to a function which is finite everywhere and belongs to L in every finite range, then by Offord's theorems 8 and 5, we obtain that the function

$$\frac{f(\mu,x)}{i\varepsilon x}(e^{i\varepsilon x}-1)$$

belongs to L_p^* and $F_\varepsilon(\mu+it)$ belongs to H^p.

Then by the use of the previous theorem we have

$$F_\varepsilon(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\mu+iy)}{\mu+iy-s}dy-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(-\lambda+iy)}{-\lambda+iy-s}dy, \quad -\lambda<\sigma<\mu.$$

On account of the fact that both of the integrals converges uniformly in ε, let ε tend to zero we have

$$F(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\mu+iy)}{\mu+iy-s}dy-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(-\lambda+iy)}{-\lambda+iy-s}dy.$$

Thus, together with the conditions that $F(\mu+it)$ and $F(-\lambda+it)$ belong to H^pL^p, we obtain the conclusions of Theorems 4 and 5.

§4. The Fourier transform of a function in a half-plane

THEOREM 8 *If $F(\sigma+it)$ is analytic for $\sigma \geqslant 0$ and $F(it)$ belongs to H^p, and if*

$$\int_{-\infty}^{\infty} |F(\sigma+it)|^p dt < \text{const.}$$

for $0 \leqslant \sigma < \infty$, then the formula

$$F(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(iy)}{iy-s}$$

holds for $R(s) > 0$. Moreover, there exists a function $f(x)$ which vanishes almost everywhere for $x > 0$ and which belongs to $L^p(-\infty, 0)$ such that for $\sigma > 0$

$$F(\sigma+it) = \int_{-\infty}^{0} f(x)e^{x(\sigma+it)}dx,$$

where the integral converges everywhere in t in the ordinary sense. $F(\sigma+it)$ and $f(x)e^{\sigma x}$ are Fourier transforms of each other in L^p for each $\sigma > 0$, and both of them belong to H^pL^p.

Proof By theorem 3, we establish the formula

$$F(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(iy)}{iy-s}dy.$$

since

$$\left|\int_{-\infty}^{\infty} \frac{F(\mu+iy)}{\mu+iy-s}dy\right| \leqslant \text{const.}\left\{\int_{-\infty}^{\infty} \frac{dy}{|\mu+iy-s|^\rho}\right\}^{\frac{1}{q}} = o(1)$$

as $\mu \to \infty$.

Now, just as we have done on theorem 5, we can prove that

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f(0,x)e^{\sigma x+itx}dx,$$

where $f(0,x)$ is the Fourier transform of $F(it)$ in L^p.

We define

$$f(x) = \begin{cases} f(0,x) & (x<0), \\ 0 & (x>0). \end{cases}$$

Therefore

$$\int_{-\infty}^{\infty} |f(x)|^p dx = \int_{-\infty}^{0} |f(0,x)|^p dx < \infty.$$

And by Theorems 1 and 2, $f(x)e^{\sigma x}$ belongs to H^pL^p. So is $F(\sigma+it)$ for each $\sigma > 0$.

THEOREM 9　*If $F(s)$ is analytic over $0 \leqslant \sigma < \infty$, if*

$$|F(s)| < M$$

uniformly over $\sigma \geqslant 0$, and if $F(it)$ belongs to H^pL^p, then the conclusions of Theorem 8 are valid.

Proof　Just as we have done before, for $\sigma > 0$ we have

$$F(s) = \frac{1}{2\pi} \lim_{B\to\infty} \int_{-B}^{B} \frac{F(\mu+iy)}{\mu+iy-s} dy - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(iy)}{iy-s} dy, \tag{4.1}$$

where μ is any number greater than σ. Let us consider

$$\begin{aligned} &\frac{1}{2\pi i} \int_{B}^{B+1} dA \left[\int_{-Ai}^{\mu-Ai} + \int_{\mu-Ai}^{\mu+Ai} + \int_{\mu+Ai}^{Ai} + \int_{Ai}^{i(t+\rho)} + \int_{i(t-\rho)}^{-Ai} \right] \frac{F(z)}{z-it} dz \\ &+ \frac{1}{2\pi i} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\rho e^{\theta i} + it) i d\theta = 0, \end{aligned} \tag{4.2}$$

where $\rho e^{\theta i} = z - it$. In the first place,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\rho e^{\theta i} + it) d\theta &= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(it) dt + \frac{\rho}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{F(\rho e^{i\theta} + it) - F(it)}{\rho e^{i\theta}} e^{\theta i} d\theta \\ &= \frac{F(it)}{2} + O(\rho) \end{aligned}$$

as ρ tends to zero since $F(z)$ is analytic at $z = it$.

As before, let B tend to infinity then (4.2) becomes

$$\begin{aligned} &\frac{1}{2\pi} \left\{ \lim_{B\to\infty} \int_{-B}^{B} \frac{F(\mu+iy)}{\mu+iy-it} dy - \left[\int_{-B}^{t-\rho} + \int_{t+\rho}^{B} \right] \frac{F(iy)}{i(y-t)} dy \right\} \\ &+ \frac{1}{2} F(iy) + O(\rho) = 0. \end{aligned} \tag{4.3}$$

Since $F(iy)$ belongs to L^p, by a theorem due to Riesz⑤, when ρ tends to zero, the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(iy)}{y-t} dy$$

converges almost everywhere in t to a function $g(t)$ which belongs to L^p (in the sense of the principal value of Cauchy). Again,

$$\lim_{B\to\infty} \int_{-B}^{B} \left[\frac{F(\mu+iy)}{\mu+iy-s} - \frac{F(\mu+it)}{\mu+iy-it} \right] dy \tag{4.4}$$

⑤ M. Riesz, Sur les fonctions conjuguées. *Math. Zeits.*, 1928, 27: 218–244.

$$\leqslant \text{const.} \int_{-\infty}^{\infty} \frac{dy}{|\mu + iy - s||\mu + iy - it|} = o(1)$$

as μ tends to infinity.

Together with (4.1), (4.3) and (4.4), we obtain

$$\begin{aligned} F(\sigma + it) &= \text{const.} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(iy)dy}{iy - \sigma - it} \\ &= g(t) - \frac{1}{2}F(it) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(iy)}{iy - \sigma - it} dy. \end{aligned} \tag{4.5}$$

Because $g(t) - F(it)$ belongs to L^p, the constant in (4.5) must vanish and we have the formula

$$F(\sigma + it) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(iy)}{iy - \sigma - it} dy.$$

The other parts of the conclusion follow immediately as we have indicated in the preceding theorem.

By an argument similar to the one we have used in theorem 7, we obtain the following theorem of Phragmén-Lindelöf type:

THEOREM 10 *If $F(s)$ is analytic over $0 \leqslant \sigma < \infty$, if*

$$\lim_{r \to \infty} \frac{1}{r} \log |F(re^{\theta i})| = 0$$

and if $F(it)$ belongs to $H^p L^p$, then the conclusions of Theorem 8 are valid.

§5. Entire functions of exponential type

THEOREM 11 *The two following classes of entire functions are identical:*

(1) *the class of entire functions $F(z)$ belonging to $H^p L^p$ along real axis and satisfying the condition*

$$F(z) = O\left(e^{A|z|}\right);$$

(2) *the class of all entire functions of the form*

$$F(z) = \int_{-A}^{A} f(u) e^{iuz} du.$$

where $f(u)$ belongs to L^p over $(-A, A)$.

Proof The class (2) is contained in the class (1). In the first place, $F(x)$ belongs to $H^p L^p$ by Theorems 1 and 2. In the second place

$$|F(z)| \leqslant \left\{ \int_{-A}^{A} |f(u)|^p du \right\}^{\frac{1}{p}} \left\{ \int_{-A}^{A} |e^{iquz}| du \right\}^{\frac{1}{q}}$$

$$=\text{const.}\left\{\int_{-A}^{A} e^{qu|Iz|}du\right\}^{\frac{1}{q}}$$

$$=\text{const.}\left\{\frac{e^{qA|Iz|}-e^{-qA|Iz|}}{q|Iz|}\right\}^{\frac{1}{q}}$$

$$=O\left(e^{A|z|}\right).$$

On the other hand, the class of (1) is of the form (2). In order to prove this, let us consider the function

$$G(z)=\frac{e^{-Az}}{\varepsilon}\int_{z}^{z+i\varepsilon} F(iw)dw,$$

which is bounded over the imaginary axis and positive real axis, and which is at most of exponential growth. By the Phragmén-Lindelöf theorem, it is bounded on the right half-plane. As we have done for Theorem 7, $G(it)$ belongs to H^pL^p. Hence by theorem 8 there exists $f_\varepsilon(x)$ belonging to $L^p(-\infty,0)$ such that

$$G(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f_\varepsilon(x)e^{zx}dx \quad (R(z)>0) \tag{5.1}$$

and $G(z)$ belongs to H^pL^p for each $x=R(z)>0$. Thus we have

$$F_\varepsilon(z)=\frac{1}{\varepsilon}\int_{z}^{z+\varepsilon} F(w)dw=iG(iz)e^{iAz}$$
$$=\int_{-\infty}^{A} if_\varepsilon(x-A)e^{izx} \quad (I(z)<0).$$

Similarly, we obtain

$$\frac{1}{\varepsilon}\int_{z}^{z+\varepsilon} F(w)dw=\int_{-A}^{\infty} i\overline{f}_\varepsilon(x+A)e^{izx}dx \tag{5.2}$$

for $I(z)>0$.

Let $H(z)=F_\varepsilon(-iz)$. Then

$$\begin{aligned} H(z)&=\int_{-A}^{\infty} i\overline{f}_\varepsilon(x+A)e^{zx}dx \quad (R(z)>0),\\ &=\int_{-\infty}^{A} if_\varepsilon(x-A)e^{zx}dx \quad (R(z)<0). \end{aligned} \tag{5.3}$$

Since $F(z)$ is at most of exponential growth, so is $H(z)$; because of (5.3), $H(z)$ belongs to H^pL^p for each $\sigma=R(z)\neq 0$. Thus by Theorem 7 $H(it)$ belongs to H^pL^p

in t and there exists a function $g_\varepsilon(x)$ such that

$$\frac{1}{\varepsilon}\int_z^{z+\varepsilon} F(w)dw = \int_{-\infty}^{\infty} g_\varepsilon(u)e^{izu}du \quad (\mathrm{C}, 1) \tag{5.4}$$

for any z. Comparing with (2.4) and (5.3) we have

$$g_\varepsilon(x) = \begin{cases} i\overline{f}_\varepsilon(x+A) & \text{for} \quad -A \leqslant x < 0, \\ if_\varepsilon(x-A) & \text{for} \quad A \geqslant x > 0, \\ 0 & \text{for} \quad |x| > A. \end{cases}$$

In particular,

$$\frac{1}{\varepsilon}\int_x^{x+\varepsilon} F(w)dw = \int_{-A}^{A} g_\varepsilon(u)e^{ixu}du.$$

On the other hand since $F(x)$ belongs to H^pL^p, we have

$$F(x) = \int_{-\infty}^{\infty} f(u)e^{ixu}du \qquad (\mathrm{C}, 1)$$

and

$$\frac{1}{\varepsilon}\int_x^{x+\varepsilon} F(x)dx = \int_{-\infty}^{\infty} f(u)\frac{e^{i\varepsilon u}-1}{i\varepsilon u}e^{ixu}du.$$

Hence comparing with (5.4) $f(u)$ must vanish outside $(-A, A)$. Therefore, as ε tends to zero, (5.4) becomes

$$F(z) = \int_{-A}^{A} f(u)e^{izu}du.$$

An immediate corollary is

THEOREM 12 *If $F(z)$ is an entire function such that*

$$\lim_{r\to\infty}\frac{1}{r}\overset{+}{\log}|F(re^{i\theta})| = 0$$

and does not vanish identically, it can not belong to H^pL^p along any line.

If it does, we may take this line to be the real axis. Thus for every $A > 0$, the Fourier transform of $F(x)$ will vanish almost everywhere outside $(-A, A)$ by the preceding theorem. Hence $F(z)$ must vanish identically and this contradicts our hypothesis.

A REMARK ON THE MOMENT PROBLEM

LOO-KENG HUA*

Fox's results† on the moment problem can be generalized in the following way.

Let (a, b) be a finite or infinite interval, and let $p(t)$ be a real-valued function such that $t^s p(t)$ is summable in $(a, b)(s = 0, 1, \cdots, 2n)$ for some $n \geqslant 0$. Let $\{P_i(t)\}(i = 0, 1, \cdots, n)$ be a set of polynomials with real coefficients such that (i) the degree of any $P_i(t)$ is at most n, and (ii)

$$\int_a^b p(t)P_i(t)P_j(t)dt = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

If $p(t)$ is non-negative and not equivalent to zero, such a set of polynomials always exists.

Let

$$P_i(t) = \sum_{j=0}^{n} a_{ij}t^j, \quad (i = 0, 1, \cdots, n).$$

We note that the polynomials $P_i(t)$ are linearly independent. For, if

$$\sum_{j=0}^{n} \lambda_j P_j(t) = 0,$$

then

$$\sum_{j=0}^{n} \lambda_j \int_a^b p(t)P_i(t)P_j(t)dt = 0, \quad (i = 0, 1, \cdots, n),$$

i.e., $\lambda_i = 0(i = 0, 1, \cdots, n)$. It follows that the matrix (a_{ij}) is non-singular and has an inverse matrix (b_{ij}), so that

$$t^i = \sum_{j=0}^{n} b_{ij}P_j(t), \quad (i = 0, 1, \cdots, n).$$

THEOREM *For any set of numbers $\{c_r\}(r = 0, 1, \cdots, n)$, the system of equations*

$$\int_a^b p(t)f(t)t^r dt = c_r, \quad (r = 0, 1, \cdots, n) \tag{1}$$

* Received 31 November, 1938; read 19 *January, 1939. Extracted from the Journal of the London Mathematical Society*, 1939, 14.

† C. Fox. The solution of a moment problem, *Journal London Math. Soc.*, 1938, 13: 12–14.

has one and only one solution of the form

$$f(t) = \sum_{i=0}^{n} e_i P_i(t).$$

viz., that given by

$$e_i = \sum_{j=0}^{n} a_{ij} c_j, \quad (i = 0, 1, \cdots, n).$$

(i) Suppose that

$$f(t) = \sum_{i=0}^{n} e_i P_i(t)$$

is a solution of (1). Then

$$\sum_{j=0}^{n} a_{ij} c_j = \sum_{j=0}^{n} a_{ij} \int_a^b p(t) f(t) t^j dt = \int_a^b p(t) f(t) P_i(t) dt$$
$$= \sum_{k=0}^{n} e_k \int_a^b p(t) P_k(t) P_i(t) dt = e_i, \quad (i = 0, 1, \cdots, n).$$

(ii) Suppose that

$$f(t) = \sum_{i=0}^{n} e_i P_i(t).$$

where

$$e_i = \sum_{j=0}^{n} a_{ij} c_j, \quad (i = 0, 1, \cdots, n).$$

Then

$$\int_a^b p(t) f(t) t^r dt = \sum_{i,j,k=0}^{n} b_{rk} a_{ij} c_j \int_a^b p(t) P_i(t) P_k(t) dt$$
$$= \sum_{i,j=0}^{n} b_{ri} a_{ij} c_j = c_r, \quad (r = 0, 1, \cdots, n).$$

i.e., $f(t)$ is a solution of (1).

The following examples may be mentioned.

(I) If $p(t) = 1, a = 0, b = 1$, we have Fox's result.

(II) Let $p(t) = e^{-t}, a = 0, b = +\infty$. If $L_n(t)$ is the Laguerre polynomial of degree n, we can take

$$P_i(t) = \frac{1}{i!} L_i(t) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \frac{1}{j!} t^j.$$

Our results then state that the only solution $f(t)$ of

$$\int_0^\infty e^{-t} f(t) t^r dt = c_r, \quad (r = 0, 1, \cdots, n),$$

such that $f(t)$ is a polynomial of degree not more than n, is

$$f(t) = \sum_{i=0}^{n} \left\{ \sum_{j=0}^{n} (-1)^j \binom{i}{j} \frac{1}{j!} c_j \right\} \frac{1}{i!} L_i(t).$$

Many other examples can easily be constructed.

National Tsing Hua University,

Kunming,

Yunnan, China.

ESTIMATION OF AN INTEGRAL*

Let $\omega(u)$ be a real function for $u \geqslant 1$, defined by

$$\begin{cases} \omega(u) = u^{-1}, & 1 \leqslant u \leqslant 2; \\ \dfrac{d}{du}(u\omega(u)) = \omega(u-1), & u > 2. \end{cases} \tag{1}$$

In 1937, the Soviet mathematician Buchstab[1] estimated

$$\lim_{u\to\infty} \omega(u) = e^{-\gamma}, \tag{2}$$

where γ is the Euler constant. This result is a consequence of his work on number theory. His original proof was obtained by Brun's method in number theory. Evidently, this is purely an analytical problem, since the definition of $\omega(u)$ and the conclusion are not related to number theory. Thus we ask whether we can use an analytic method to prove this proposition. In 1950, the Dutch mathematician De Bruijn[2] achieved this. But in 1951, Buchstab proved more precisely that

$$|\omega(u) - e^{-\gamma}| < e^{-u(\log u + \log\log u - 1) + O(u \log\log u/\log u)}. \tag{3}$$

by a number-theoretical method.

In this paper I use an analytical method to prove a result more precise than (3).

Lemma 1 *Let*

$$g(x) = \exp\left(-x + \int_0^x \frac{e^{-t}-1}{t} dt\right). \tag{4}$$

This function has the properties

(i) $g(x) = \dfrac{d}{dx}(xe^x g(x))$;

(ii) $\displaystyle\int_0^\infty g(x)dx = e^{-\gamma}$.

Proof Differentiating

$$\begin{aligned} \frac{d}{dx}(xe^x g(x)) =& \frac{d}{dx}\left(x \exp \int_0^x \frac{e^{-t}-1}{t} dt\right) \\ =& \exp\left(\int_0^x \frac{e^{-t}-1}{t} dt\right)\left(1 + x\frac{e^x - 1}{x}\right) = g(x). \end{aligned}$$

$*$ Published in *Sci Sinica*, 1951, 4: 393–402.

Integrating from 0 to ∞,

$$\int_0^\infty g(x)dx = [xe^x g(x)]_0^\infty = \left[x\exp\left(\int_0^x \frac{e^{-t}-1}{t}dt\right)\right]_0^\infty$$
$$= \lim_{x\to\infty} \exp\left(\int_0^x \frac{e^{-t}-1}{t}dt + \log x\right) = e^{-\gamma}.$$

Lemma 2 *Let*

$$h(u) = \int_0^\infty g(x)e^{-ux}dx. \tag{5}$$

This integral is absolutely convergent for $u \geqslant -1$, and uniformly convergent for $u \geqslant -1+\varepsilon$. Also, $h(u)$ has the following properties:

(i) $h(0) = e^{-\gamma}$;

(ii) $\lim\limits_{u\to\infty} uh(u) = 1$;

(iii) $uh'(u-1) + h(u) = 0, \quad$ *for* $u > 0$;

(iv) $uh(u-1) + \int_{u-1}^u h(t)dt = 1, \quad$ *for* $u > 0$.

Proof (i) can be obtained from Abel's Lemma and Lemma 1, (ii). Putting $ux = y$, then

$$uh(u) = \int_0^\infty g\left(\frac{y}{u}\right)e^{-y}dy,$$

and (ii) follows.

Differentiating under the sign of integration, by (5),

$$uh'(u-1) = -u\int_0^\infty g(x)e^{-(u-1)x}xdx$$
$$= \int_0^\infty xe^x g(x)d\left(e^{-ux}\right)$$
$$= xe^x g(x)e^{-ux}\Big|_0^\infty - \int_0^\infty e^{-ux}d(xe^x g(x))$$
$$= -\int_0^\infty e^{-ux}g(x)dx = -h(u).$$

(Lemma 1, (i)), and (iii) are proved.

Now we prove

$$q(u) = uh(u-1) + \int_{u-1}^u h(t)dt$$

is a constant. Differentiating $q(u)$, from (iii) we have

$$q'(u) = h(u-1) + uh'(u-1) + h(u) - h(u-1) = 0,$$

So $q(u)$ is a constant. Let $u \to \infty$. By (ii) we have

$$\lim_{u \to \infty} q(u) = 1.$$

Therefore (iv) is proved.

Lemma 3 *For $u \geqslant 2$, we have the identity*

$$\int_{u-1}^{u} \omega(t)h(t)dt + u\omega(u)h(u-1) = e^{-\gamma}.$$

Proof Denote the left-hand side by $p(u)$. By (1) and Lemma 2 (ii)

$$p'(u) = \omega(u)h(u) - \omega(u-1)h(u-1) + \frac{d}{du}(u\omega(u))h(u-1) + u\omega(u)h'(u-1) = 0,$$

so $p(u)$ is a constant. By (1), we have

$$\begin{aligned} p(2) &= \int_{1}^{2} \omega(t)h(t)dt + 2\omega(2)h(1) \\ &= \int_{1}^{2} \frac{h(t)}{t}dt + h(1) \\ &= -\int_{1}^{2} h'(t-1)dt + h(1) \qquad \text{(by Lemma2(ii))} \\ &= h(0) = e^{-\gamma} \qquad \text{(by Lemma1(i))} \end{aligned}$$

Lemma 4 *Let*

$$W(u) = \omega(u) - e^{-\gamma}, \tag{6}$$

then for $u > 0$ we have

$$W(u) = -\frac{1}{uh(u-1)} \int_{u-1}^{u} W(t)h(t)dt. \tag{7}$$

Proof By Lemma 3 and Lemma 2 (iv),

$$\int_{u-1}^{u} \omega(t)h(t)dt + u\omega(u)h(u-1) = e^{-\gamma}\left(\int_{u-1}^{u} h(t)dt + uh(u-1)\right),$$

Hence

$$\int_{u-1}^{u} W(t)h(t)dt + uh(u-1)W(u) = 0.$$

This is (7).

Let

$$F(u) = |W(u)|.$$

By (7), we have

$$F(u) \leqslant \frac{1}{uh(u-1)} \int_{u-1}^{u} F(t)h(t)dt$$
$$= \frac{1}{uh(u-1)} \int_{0}^{1} F(u-1+\vartheta)h(u-1+\vartheta)d\vartheta$$
$$\leqslant \frac{1}{u} \int_{0}^{1} F(u-1+\vartheta)d\vartheta \tag{8}$$

(since $h(u)$ is a monotone decreasing function).

Lemma 5 *Suppose $f(u)$ is a positive function, and $f(u)$ satisfies* (8) *for sufficiently large u, then, as $u \to \infty$, we have*

$$F(u) \leqslant e^{-u(\log u + \log\log u + \log\log u/\log u - 1) + O(u/\log u)}.$$

Proof (为了使读者易于了解起见, 在证明中明确地引入"逐步求精法". 并且避免引用 Stirling 公式之类, 不然这证明是可以大大地缩短的). Let $M(u) = \max_{u \leqslant x \leqslant \infty} F(x)$. From (8) we have

$$M(u) \leqslant \frac{M(u-1)}{u} \leqslant \frac{M(u-2)}{u(u-1)} \leqslant \cdots = O\left(\frac{1}{P(u)}\right),$$

where

$$\log P(u) = \log u + \log(u-1) + \cdots.$$

Since $\log x$ is an increasing function,

$$\log P(u) \geqslant \int_{1}^{u} \log x dx = u(\log u - 1).$$

Hence

$$M(u) = O\left(e^{-u(\log u - 1)}\right). \tag{9}$$

2. Let

$$F_1(u) = F(u)e^{u(\log u - 1)}$$

and

$$M_1(u) = \max_{u \leqslant x \leqslant \infty} F_1(u).$$

From (8) we immediately obtain

$$M_1(u) \leqslant \frac{M_1(u-1)}{u} \int_0^1 \frac{e^{u(\log u - 1)}}{e^{(u-1+t)(\log(u-1+t)-1)}} dt$$

$$=\frac{M_1(u-1)}{u}\int_0^1 \exp(\Phi(t))dt. \tag{10}$$

where

$$\begin{aligned}\Phi(t) &= u(\log u-1)-(u+t-1)(\log(u+t-1)-1)\\ &\leqslant u(\log u-1)-(u+t-1)(\log(u-1)-1)\\ &= \log u-1+(u-1)(\log u-\log(u-1))-t(\log(u-1)-1)\\ &\leqslant \log u-t(\log(u-1)-1).\end{aligned}$$

Substituting in (10), we have

$$M_1(u)\leqslant M_1(u-1)\int_0^1 e^{-t(\log(u-1)-1)}dt \leqslant \frac{M_1(u-1)}{\log(u-1)-1}.$$

Successively using this expression, we obtain

$$M_1(u)=O\left(\frac{1}{P_1(u)}\right),$$

where

$$\begin{aligned}\log P_1(u) &= \log(\log(u-1)-1)+\log(\log(u-2)-1)+\cdots\\ &\geqslant \int^{u-1}\log(\log t-1)dt\\ &= \left[t\log(\log t-1)\int\frac{dt}{\log t-1}\right]^{u-1}\\ &= u\log\log u - C_1\frac{u}{\log u},\end{aligned}$$

and C_1 is a constant > 1.

Now we have proved

$$F(u)=O\left(e^{-u(\log u+\log\log u-1)+C_1u/\log u}\right). \tag{11}$$

This result is more accurate than Buchstab's, but we can get a much better result.

3. Let

$$F_2(u)=F_1(u)e^{u\log\log u-C_1u/\log u},$$

and

$$M_2(u)=\max_{u\leqslant x\leqslant\infty}F_2(x).$$

From (8) we immediately obtain

$$M_2(u) \leqslant \frac{M_2(u-1)}{u} \int_0^1 \exp(\Phi(t))dt, \tag{12}$$

where

$$\begin{aligned}
\Phi(t) - \log u =& -\log u + u(\log u + \log\log u - 1) - C_1 u/\log u - (u+t-1) \\
& \times (\log(u+t-1) + \log\log(u+t-1) - 1) + C_1 \frac{u+t-1}{\log(u+t-1)} \\
\leqslant& (u-1)\log u + u\log\log u - C_1 u/\log u \\
& - (u+t-1)(\log(u-1) + \log\log(u-1) - 1) + C_1 \frac{u+t-1}{\log(u-1)} \\
=& (u-1)\log\frac{u}{u-1} + \log\log u + (u-1)\log\left(\frac{\log u}{\log(u-1)}\right) \\
& - 1 - C_1\left(\frac{u}{\log u} - \frac{u-1}{\log(u-1)}\right) \\
& - t\left(\log(u-1) + \log\log(u-1) - 1 - \frac{C_1}{\log(u-1)}\right) \\
\leqslant& \log\log u + (u-1)\frac{1}{u-1} + (u-1)\frac{1}{(u-1)\log(u-1)} - 1 \\
& - \frac{C_1}{\log(u-1)} - t\left(\log u + \log\log u - 1 - \frac{C_1}{\log(u-1)}\right) \\
& \qquad (\text{since } \log(1+x) \leqslant x) \\
\leqslant& \log\log u - t(\log u + \log\log u - 2) + \frac{1}{\log(u-1)} \\
& \qquad (\text{for sufficiently large } u).
\end{aligned}$$

Substituting in (12), we have

$$\begin{aligned}
M_2(u) \leqslant& M_2(u-1)\frac{e^{1/\log(u-1)}\log u}{\log u + \log\log u - 2} \\
\leqslant& M_2(u-1)\exp\left(-\log\left(1 + \frac{\log\log u}{\log u} - \frac{2}{\log u}\right) + \frac{1}{\log(u-1)}\right) \\
\leqslant& M_2(u-1)\exp\left(-\frac{\log\log u}{\log u} + \frac{C_2}{\log u}\right)
\end{aligned}$$

(since $\log(1+x) \geqslant x - Cx^2$). But

$$\begin{aligned}
& \frac{\log\log u}{\log u} - \frac{C_2}{\log u} + \frac{\log\log(u-1)}{\log(u-1)} - \frac{C_2}{\log(u-1)} + \cdots \\
\geqslant& \int^{u-1} \frac{\log\log t}{\log t}dt - C_2\int^u \frac{dt}{\log t} \\
\geqslant& \frac{u\log\log u}{\log u} - C_3\frac{u}{\log u}.
\end{aligned}$$

So we obtain

$$M_2(u) \leqslant e^{u \log\log u/\log u - C_3 u/\log u},$$

that is

$$F(u) \leqslant e^{-u(\log u + \log\log u + \log\log u/\log u - 1) + C_4 u/\log u}.$$

Remark We can iterate this method, so as to obtain even sharper results.

Combining Lemma 5, Lemma 4 and (6), we have

Theorem $|\omega(u) - e^{-\gamma}| \leqslant e^{-u(\log u + \log\log u + \log\log u/\log u - 1) + O(u/\log u)}$.

If it suffices to prove merely

$$\lim_{u\to\infty} \omega(u) = e^{-\gamma}.$$

Then we have the following simple method, but need to quote some theorems on the Laplace integral. Let

$$f(s) = \int_0^\infty e^{-us} d\alpha(u) = \int_0^\infty e^{-us} dw(u+1), \tag{13}$$

where

$$\alpha(u) = \omega(u+1) - 1.$$

Integrating by parts

$$\begin{aligned} f(s) =& s\int_0^\infty e^{-us}\alpha(u)du = -1 + s\int_0^\infty e^{-us}\omega(u+1)du \\ =& -1 + s\int_0^\infty e^{-us} d((u+2)w(u+2)). \end{aligned}$$

Changing u to $u-1$, and using (1) we have

$$\begin{aligned} f(s) =& -1 + se^s \int_0^\infty e^{-us} d((u+1)\omega(u+1)) \\ =& -1 + se^s \left[\int_0^\infty e^{-us}(u+1)d\omega(u+1) + \int_0^\infty e^{-us}\omega(u+1)du\right]. \end{aligned} \tag{14}$$

Differentiating under the integral we have

$$\int_0^\infty ue^{-us} d\omega(u+1) = -f'(s),$$

and integrating by parts

$$\int_0^\infty e^{-us}\omega(u+1)du = -\frac{e^{-us}}{s}\omega(u+1)\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-us} d\omega(u+1)$$

$$=\frac{1}{s}+\frac{1}{s}f(s).$$

Putting these two expressions into (14), we have

$$f(s)=-1+se^{s}\left[f(s)-f'(s)+\frac{1}{s}+\frac{1}{s}f(s)\right],$$

i.e. the ordinary differential equation of first order

$$f'(s)+\left(s^{-1}\left(e^{-s}-1\right)-1\right)f(s)=s^{-1}\left(1-e^{-s}\right). \tag{15}$$

Since $\lim_{u\to 0}\alpha(u)=\lim_{u\to 0}(\omega(u+1)-1)=0$, it follows from Abel's Theorem (ex. [4], p. 183, Cor. 1c), that

$$\lim_{s\to\infty}f(s)=0. \tag{16}$$

Solving the differential equation (15) and using condition (16), we obtain

$$\begin{aligned}
f(s)=&-e^{s+\int_0^s t^{-1}(1-e^{-t})dt}\int_s^{\infty}e^{-u-\int_0^u t^{-1}(1-e^{-t})dt}\frac{1-e^{-u}}{u}du\\
=&\,e^{s+\int_0^s t^{-1}(1-e^{-t})dt}\int_s^{\infty}e^{-u-\int_0^u t^{-1}(1-e^{-t})dt}d\left(-u-\int_0^u t^{-1}(1-e^{-1})dt\right)\\
&+e^{s+\int_0^s t^{-1}(1-e^{-t})dt}\int_s^{\infty}e^{-u-\int_0^u t^{-1}(1-e^{-t})dt}du\\
=&-1+e^{s+\int_0^s t^{-1}(1-e^{-t})dt}\int_s^{\infty}e^{-u-\int_0^u t^{-1}(1-e^{-t})dt}du\\
=&-1+e^{s+\int_0^s t^{-1}(1-e^{-t})dt}[ue^{-\int_0^u t^{-1}(1-e^{-t})dt}]_s^{\infty}\\
=&-1-se^{s}+e^{-\gamma+s+\int_0^s t^{-1}(1-e^{-t})dt}.
\end{aligned} \tag{17}$$

It is clear that

$$\lim_{s\to 0}f(s)=-1+e^{-\gamma}.$$

Then, by a Tauberian Theorem, we obtain

$$\lim_{u\to\infty}\alpha(u)=\lim_{u\to\infty}(\omega(u+1)-1)=-1+e^{-\gamma}.$$

This is

$$\lim_{u\to\infty}\omega(u)=e^{-\gamma}.$$

But note that it is necessary to check the Tauberian Condition, i.e. $\int_0^t u\,d\alpha(u)=O(t)$. (ex. [4], p. 187, Thm. 36). Since

$$d\omega(u)=\frac{1}{u}(\omega(u-1)-\omega(u))du,$$

then

$$\int_0^t u d\alpha(u) = \int_0^t u d\omega(u+1) = \int_1^t [\omega(u) - \omega(u+1)]du + O(1)$$
$$= -\int_t^{t+1} \omega(u)du + O(1) = O(1)$$

(it is easy to prove $\omega(u) = O(1)$). So the Tauberian Condition is satisfied. The Theorem is now completely proved.

Note that, incidentally, we have found (17) to be the Laplace transform of $\omega(u+1)$.

Summary

Let $\omega(u)$ be the function defined by (1) and (2), $f(s)$ be the Laplace transform of $\omega(u+1)$. We proved that $f(s)$ satisfies the differential equation (14) with the initial conditon $\lim_{s\to\infty} f(s) = 0$. Solving the differential equation, we obtain the explicit expression (17). By Tauberian theorem, we deduce that $\lim_{u\to\infty} \omega(u) = e^{-\gamma}$, where γ is the Euler constant. Further from (2), we have

$$\omega'(u) = -\frac{1}{u}\int_{u-1}^{u} \omega^t(t)dt,$$

we deduce then

$$\omega'(u) = O\left(e^{-u(\log u + \log\log u + \log\log u/\log u - 1) + O(u/\log u)}\right).$$

Integrating, we have

$$\omega(u) = e^{-\gamma} + O\left(e^{-u(\log u + \log\log u + \log\log u/\log u - 1) + O(u/\log u)}\right).$$

which is sharper than a result due to Buchstab and answers a conjecture of De Bruijn. The proof in the text requires no knowledge deyond advanced calculus.

References

[1] Buchstab, A.A. An asymptotic estimation of a general number-theoretic function. *Matem Sb.*, 1937, 2(144): 1239–1246.

[2] DeBruijn N. G., On the number of uncancelled elements in the sieve of Eratosthenes. *Nederl. Akad. Wetensch. Proc.*, 1950, **52**: 803–812; *Indagationes Mathematicae,* **12**: 247–256.

[3] Buchstab A. A. On the estimation of the number of numbers in an arithmetical progression which cannot be divided by "relatively" small primes. *Matem. Sb.*, 28(70): 166–184.

[4] Widder D. V. The Laplace Transform.

广义函数导引*

华罗庚

这篇文章完成于 1957 年, 曾在中国科学院数学研究所的讨论班上报告过. 因为创造性的成果不多, 一直没有发表. 最近感谢关肇直同志告诉我 Bremermann 及 Durand(*Math. Physics*, 1961) 也有相似的看法 (肇直源于 Köthe, *Math. Zeits.*, 1952, 57: 13-33). 他们把 S 类广义函数 (实质上, 他们应当跳出 L. Schwartz 的限制, 而研究本文中的更广泛的 H 类的) 看成为可以用两个解析函数来表达的对象. 扼要地说: 圆内解析函数在圆上是只有"正"项的 Fourier 级数, 而圆外解析函数是在圆上只有"负"项的 Fourier 级数. 合起来才能表示一个"完整的" Fourier 级数. 我们这儿所说的 H 类 (当然包有 S 类) 函数看成为圆内一个调和函数边界值, 它本身就是"完整的" Fourier 级数. 看上去这方法还有发展的前途, 因而仍照 1957 年原油印稿发表于此.

对于关肇直同志、龚昇同志、陆启铿同志的帮助和提意见在此表示感谢.

§1. 引　　言

本文之目的在于给出广义函数论的一个简单而浅近的介绍. L. Schwartz[2] 所研究的广义函数类仅是本文所论的很多广义函数类的一个而已, 我们所获得类比 Schwartz 的更有许多好处. 本文是自给自足的. 任何了解一致收敛的读者都可以没有很大的困难来了解本文, 但是初学者, 最好先看后面的附录, 并且把本文中描述性的部分略去 ①.

我们仅讨论一个变数的情况, 但是对娴熟技巧的读者很可能自己把这些结果推广到 Schwartz 所研究过的多变数情况. 但是本文中还将建议以下几种更有趣的多变数的推广: (i) 把 Laplace 方程换为任一椭圆型偏微分方程及 (ii) 把这些研究推广到酉群上的广义函数论, 因为与酉群有关的 Dirichlet 问题已为作者 [4] 所解决了. 最有趣的一点是我们这个具体的初等的处理法反而比 Schwartz 的抽象的处理法建议得更多.

广义函数的主要看法可以以下的一简单的叙述概括之:

---

* 关于广义函数的来源见Н.М. Гюнтер及 С.Л. Соболев(参考И.М. Гельфанд-Г.Е.Щ.нлов[1]). 1962 年 9 月 7 日收到.

① 在 [3] 中有详尽的参考文献.

"对应于一个调和函数, 我们有一个边界值函数 (可能就是一个存在的普通函数, 或可能是一个虚拟的函数), 这就叫做广义函数. 由于调和函数是无穷可微的, 因此广义函数也是无穷可微的".

还有另外一些看法将于本文之末加以说明.

§ 2. 定　　义

一个形式的 Fourier 级数

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

就定义为一个广义函数, 我们并不管这函数收敛与否, 这广义函数以 $u(\theta)$ 表之. 今后常用 $\sum_n$ 表 $\sum_{n=-\infty}^{\infty}$ 及 $\sum_n{}'$ 表和 $\sum_n$ 中略去 $n=0$ 一项.

命

$$v(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta},$$

显然对任二复数 λ, μ,

$$\lambda u(\theta) + uv(\theta) = \sum_{n=-\infty}^{\infty} (\lambda a_n + \mu b_n) e^{in\theta}$$

仍是一个广义函数. 故广义函数是一线性集合.

二广义函数的乘积一般是不定义的. 原因在于

$$\sum_{m+n=l} a_m b_n$$

可能不收敛.

如果级数

$$\sum_n a_n \bar{b}_n$$

收敛, 则此值称为广义函数 $u(\theta)$ 与 $\overline{v(\theta)}$ 的无向积或内积, 以 $(u(\theta), \overline{v(\theta)})$ 记之.

显然有

$$(u(\theta), \overline{v(\theta)}) = \overline{(v(\theta), \overline{u(\theta)})},$$

$$(\lambda u_1(\theta) + \mu u_2(\theta), \overline{v(\theta)}) = \lambda(u_1(\theta), \overline{v(\theta)}) + \mu(u_2(\theta), \overline{v(\theta)}),$$

且有

$$(u(\theta), \overline{e^{in\theta}}) = a_n.$$

我们定义

$$(u(\theta),\overline{v(\theta-\psi)})=(u(\theta+\psi),\overline{v(\theta)})=\sum_n a_n\bar{b}_n e^{in\psi}$$

为二函数 $u(\theta)$ 与 $v(\theta)$ 的卷积.

最有趣的例子是 Dirac 函数

$$\delta(\theta)=\sum_n e^{in\theta},$$

它使

$$(u(\theta),\overline{\delta(\theta-\psi)})=\sum_n a_n e^{in\psi}=u(\psi),$$

我们有时也把 $\delta(\theta-\psi)$ 记为 $\delta_\psi(\theta)$.

广义函数 $u(\theta)$ 的微商的定义是

$$i\sum_n na_n e^{in\theta},$$

以 $u'(\theta)$ 记之. 显然有

$$(u'(\theta),\overline{v(\theta)})=i\sum_n na_n\bar{b}_n=-\sum a_n n\overline{(ib_n)}=-(u(\theta),\overline{v'(\theta)}),$$

故立得

$$(u(\theta),\overline{\delta'(\theta-\psi)})=-(u'(\theta),\overline{\delta(\theta-\psi)})=-u'(\psi)$$

及

$$(u(\theta),\overline{\delta^{(\nu)}}(\theta-\psi))=(-1)^\nu u^{(\nu)}(\psi).$$

但是这样定义了的广义函数太广泛了, 不能得出很多的有用的结论. 我现在先引进两类特别的广义函数.

若

$$\max(\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}},\overline{\lim_{n\to\infty}}|a_{-n}|^{\frac{1}{n}})\leqslant 1,$$

则所对应的广义函数称为 H 类的广义函数, 或简称为 H 广义函数.

显然, H 类广义函数成一个线性集合, 且 H 类广义函数的微分仍是 H 类广义函数. 二 H 类的广义函数的卷积仍为 H 类函数.

若有一正整数 p 使 $a_n=O(|n|^p)$, 则所对应的广义函数称为 S 型的广义函数, 或简称 S 广义函数. 这就是 Schwartz 的广义函数.

显然, 一个 S 广义函数一定是 H 广义函数. 且 S 广义函数也是一个线性集且对微分运算及卷积运算而自封.

附记 H 类的函数间定义了两个运算"加"和"卷积". 如果把卷积视为"乘"法. 则成一环. 结合定律分配定律等等都不难直接阐明. $\lambda u(\theta)$ 也可以卷积的形式表出来:

$$\left(u(\theta), \sum_n \lambda e^{i(\psi-\theta)}\right)$$

这是一个有单位元的环, 单位元是 $\delta_0(\theta)e^{in\theta}$ 是幂等元. 这些幂等元互相正交, 其和为 $\delta_0(\theta)$.

S 类也有此同样性质.

§ 3. 对　偶　性

二类的广义函数 T 及 $\mathring{T}$ 如适合以下的三条件则称为互相对偶: (i) 对诸 $u \in T$ 及诸 $v \in \mathring{T}, (u, \bar{v})$ 常收敛, (ii) 若 $(u, \bar{v})$ 对所有的 $u \in T$ 常收敛, 则可得出 $v \in \mathring{T}$ 及 (iii) 若 $(u, \bar{v})$ 对所有的 $v \in \mathring{T}$ 常收敛, 则 $u \in T$.

例一 所有的广义函数所成的类 K 和所有的有限 Fourier 级数所成的类 $\mathring{K}$ 是对偶关系.

例二 命 $p > 1$ 及 $p' = p/(p-1)$, 则类 L^p 与 $L^{p'}$ 是对偶关系.

例三 如果把 $\sum_n a_n \bar{b}_n$ 求和法改为 $(C, 1)$ 求和法, 则还有以下一些对偶类: (i)B 与 L(此处 B 表示囿函数), (ii)C 与 St(此处 C 表示连续函数, St 表示 Fourier-Stieljes 级数所成的类). 且有关系

$$C \subset L^{\infty} = B \subset L^{p'} \subset L^2 \subset L^p \subset L \subset St^{①}.$$

定理 1 命 $\varphi(n)$ 表一递增的正函数, 当 n 趋向无穷时趋向无穷, 且假定对任一 $\delta > 0$, 级数

$$\sum_{n=1}^{\infty} \frac{1}{(\varphi(n))^{\delta}}$$

常收敛. 命 T 代表适合于

$$\log |a_n| = o(\log \varphi(|n|))$$

的广义函数类, 而 $\mathring{T}$ 是适合于

$$\log \varphi(|n|) = O\left(\log \frac{1}{|b_n|}\right)$$

的广义函数所成的类, 则类 T 与类 $\mathring{T}$ 是有对偶关系的.

① 容易证明, 如果一类广义函数, 其对偶类等于自己者, 必须是 L^2.

证　(i) 由定义, 对任一 $\varepsilon > 0$, 当 n 大时常有

$$|a_n| \leqslant \varphi(|n|)^{\varepsilon}.$$

又有一数 $c > 0$ 使

$$|b_n| \leqslant \frac{1}{(\varphi(|n|))^c},$$

故 $\sum a_n \bar{b}_n$ 是收敛的.

(ii) 假定 v 不属于 $\mathring{T}$, 则有一数列 n_v 使

$$\lim_{\nu\to\infty} \frac{\log \varphi(|n_\nu|)}{\log \dfrac{1}{|b_{n\nu}|}} = \infty.$$

取 $a_{n_\nu} = \dfrac{1}{\bar{b}_{n_\nu}}$ 及其他的 $a_n = 0$, 如此定义了一个广义函数 $u(\theta)$ 属于 T 而 $\sum a_n \bar{b}_n$ 发散.

(iii) 假定 u 不属于 T, 则有一数列 n_ν 使

$$\log |a_{n_\nu}| \geqslant c \log \varphi(|n_c|),$$

此处 $c > 0$. 取 $b_{n_\nu} = \dfrac{1}{a_{n_\nu}}$, 及其他 $b_n = 0$, 如此定义了的 $v(\theta)$ 属于 $\mathring{T}$, 且 $\sum a_n \bar{b}_n$ 发散.

在定理 1 中取 $\varphi(n) = e^n$, 则类 T 立刻变为类 H. 盖由 $\log |a_n| = o(|n|)$ 可知 $\overline{\lim\limits_{|n|\to\infty}} |a_n|^{|\frac{1}{n}|} \leqslant 1$. 又关系

$$|n| = O\left(\log \frac{1}{|b_n|}\right)$$

与次之关系等价

$$\overline{\lim_{|n|\to\infty}} |b_n|^{|\frac{1}{n}|} < 1.$$

因此得

定理 2　H 类的对偶 $\mathring{H}$ 是由适合以下的广义函数所组成的:

$$\max(\overline{\lim_{n\to\infty}} |b_n|^{\frac{1}{n}}, \overline{\lim_{n\to\infty}} |b_{-n}|^{\frac{1}{n}}) < 1.$$

取 $\varphi(n) = e^{n^p} (p > 1)$ 所得的类 T 以 G_p 表之, 由定理 1 可得

定理 3　G_p 是由适合于

$$\lim_{|n|\to\infty} |a_n|^{|n|^{-p}} \leqslant 1$$

广义函数所成的类. 它的对偶类 $\mathring{G}_p$ 是由适合于

$$\overline{\lim_{|n|\to\infty}}|b_n|^{|n|^{-p}} < 1$$

的广义函数所构成的.

类 G_p 是由Гелфанд-Шилов所引进的.

与定理 1 同法可以证明

定理 4 命 $\psi(n)$ 表一递增正函数与 n 同趋向无穷, 并假定有一正数 $\lambda > 0$ 使

$$\sum_{n=1}^{\infty}\frac{1}{(\psi(n))^{\lambda}}$$

收敛. 命 T 表适合以下条件的广义函数类

$$\log|a_n| = O(\log\psi(|n|)),$$

及 $\mathring{T}$ 为适合于

$$\log\psi(|n|) = o\left(\log\frac{1}{|b_n|}\right)$$

者. 则类 T 与类 $\mathring{T}$ 有对偶关系.

在定理 4 中取 $\psi(n) = n$. 则类 T 就是类 S, 故得

定理 5 S 类的对偶类 $\mathring{S}$ 乃由适合以下条件的广义函数所成的: 对任一 $q > 0$ 常有

$$b_n = O\left(\frac{1}{|n|^q}\right).$$

再在定理 4 中取 $\psi(n) = e^n$, 所得出的类 T 以 I 表之. 由定理 4 可得

定理 6 I 类的对偶类 $\mathring{I}$ 是由适合于

$$\lim_{|n|\to\infty}|b_n|^{-\frac{1}{|n|}} = \infty$$

的函数所组成的.

附记 类还可以分得更细. 例如, 在定理 4 中把条件换为

$$\overline{\lim_{n\to\infty}}\frac{\log|a_n|}{\log\psi(|n|)} \leqslant \rho.$$

对这种换法, 并无任何困难可求出其对偶类.

附记 极易证明, $\mathring{H}$ 类 (及 $\mathring{S}$ 类) 也是线性集合且对微分及卷积而自封. 它虽然也成一环, 但是并没有单位元. $\mathring{H}$ 是 H 的理想子. 命 $\overset{*}{H}$ 表 H 的任意一个理想子. 如果 $\overset{*}{H}$ 中所包有的广义函数都是普通函数, 也就是这广义函数的形式 Fourier 级数收敛, 则 $\overset{*}{H}$ 称为函数类的理想子. 易见, $\mathring{H}$ 是 H 的极大的函数类的理想子.

§ 4. H 广义函数的意义

对应一 H 广义函数 $u(\theta)$, 我们有一函数

$$u(r,\theta)=\sum_n a_n e^{in\theta}r^{|n|},$$

此乃一在单位圆内的调和函数 ($0\leqslant r<1$及$0\leqslant\theta\leqslant 2\pi$).

故一个 H 广义函数可以视为一个调和函数的边界值函数.

同法, 一个 $\mathring{H}$ 广义函数 $v(\theta)$(它就是一个普通意义下的函数) 对应于一个在较大同心圆中的调和函数.

又显然对应于 $\mathring{I}$ 类的函数 $v(\theta)$ 我们有一调和函数

$$\sum_n b_n e^{in\theta}r^{|n|},$$

此级数对任一 r 都收敛. 此称为调和整函数. 所以 $\mathring{I}$ 的函数就是调和整函数在单位圆上的值.

广义函数

$$\delta_\psi(\theta)$$

属于 H, 但不属于 $\mathring{H}$, 此广义函数所对应的调和函数就是

$$\sum_n e^{in(\theta-\psi)r^{|n|}}=\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2},$$

此即习知的 Poisson 核, 以 $P(r,\theta)$ 表它.

命 $f(\theta)$ 表一连续 (或可积) 函数, 其 Fourier 系数为 b_n, 则对任一 $u(\theta)\in H$ 常有

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{f(\theta)}d\theta&=\frac{1}{2\pi}\int_0^{2\pi}\sum_n a_n e^{in\theta}r^{|n|}\overline{f(\theta)}d\theta\\&=\sum_n a_n r^{|n|}\frac{1}{2\pi}\int_0^{2\pi}\overline{f(\theta)}e^{in\theta}d\theta\\&=\sum_n a_n\bar{b}_n r^{|n|}.\end{aligned}$$

若 $(u(\theta),\overline{f(\theta)})$ 收敛, 则由 Abel 定理可知

$$(u(\theta),\overline{f(\theta)})=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{f(\theta)}d\theta.$$

更一般些, 对 H 中任二广义函数 $u(\theta)$ 与 $v(\theta)$, 对 $r<1, r'<1$ 常有

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(r',\theta)}d\theta=\sum_n a_n\bar{b}_n(rr')^{|n|},$$

故若 $(u,\bar{v})$ 存在, 则

$$(u,\bar{v})=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(r,\theta)}d\theta.$$

又, 对 $u\in H$ 及 $v\in \mathring{H}$, 则有一 $\delta>0$ 使 $v(r',\theta)$ 当 $0\leqslant r'\leqslant 1+\delta$ 时调和, 取 $r=\dfrac{1}{1+\frac{1}{2}\delta}$ 及 $r'=1+\dfrac{1}{2}\delta$, 可知

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(r',\theta)}d\theta=\sum_n a_n\bar{b}_n=(u(\theta),\overline{v(\theta)}).$$

§ 5. S 广义函数的意义

任一连续函数 $u(\theta)$ 的 Fourier 系数 a_n 有次之性质

$$|a_n|\leqslant\frac{1}{2\pi}\int_0^{2\pi}|u(\theta)|d\theta=O(1).$$

作为一个 S 广义函数, 其 p 次微分的 Fourier 系数 $a_n^{(p)}$ 适合于 $a_n^{(p)}=O(|n|^p)$.

反之, 若

$$a_n=O(|n|^p),$$

则 $u(\theta)-a_0$ 是广义函数

$$\sum_n{}'\frac{a_n}{(in)^{p+2}}e^{in\theta}$$

的 $(p+2)$ 次微分. 而以上的级数一致收敛, 收敛于一个连续函数, 故类 S 的广义函数实质上就是连续函数的有限次微分.

同法可以证明: 类 $\mathring{S}$ 的广义函数就是无穷次可微分的函数.

由上节的结果已知: 若 $u\in S$, $v\in\mathring{S}$, 则

$$(u,\bar{v})=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\overline{v(\theta)}d\theta.$$

如果 $u(\theta)-a_0$ 是连续函数 $w(\theta)$ 的 p 次微分, 由分部积分法可知

$$(u,\bar{v})=a_0\bar{b}_0+\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}(u(r,\theta)-a_0)\overline{v(\theta)}d\theta$$

$$= a_0\bar{b}_0 + \lim_{r\to 1}\frac{(-1)^p}{2\pi}\int_0^{2\pi} w(r,\theta)\overline{v^{(p)}(\theta)}d\theta$$
$$= a_0\bar{b}_0 + \frac{(-1)^p}{2\pi}\int_0^{2\pi} w(\theta)\overline{v^{(p)}(\theta)}d\theta.$$

就是我们可以用普通的运算表达出来.

§ 6. 致 零 集

定义 单位圆周上的一个开区间 $a<\theta<b$ 称为一个 H-广义函数 $u(\theta)$ 的致零区间, 如果在 $a<\theta<b$ 中任一闭子区间一致地

$$\lim_{r\to 1} u(r,\theta) = 0.$$

一点 θ_0 称为 $u(\theta)$ 的支点, 如果没有包有 θ_0 的致零区间存在.

所有的 $u(\theta)$ 的致零区间的总集合称为函数 $u(\theta)$ 的致零集. 此为一开集. 其补集称为支点集. 显然支点集的任一点都是支点.

例 $\delta_\psi(\theta)$ 就是一个以 $\theta=\psi$ 为其唯一支点的广义函数, 其故为: 仅当 $\theta\neq\psi$ 时,

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}$$

趋近于 0.

定理 1 H 类中仅有 ψ 为支点的广义函数可以表示为

$$u(\theta) = \sum_{\nu=0}^{\infty} c_\nu \overline{D^{(\nu)}[\delta_\psi(\theta)]},$$

此处 $D^{(\nu)}[\delta_\psi(\theta)]$ 是 $\delta_\psi^{(0)}(\theta)=\delta_\psi(\theta), \delta'_\psi(\theta), \cdots, \delta_\psi^{(\nu)}(\theta)$ 之线性组合, 而 $\delta_\psi^{(\nu)}(\theta)$ 是 $\delta_\psi(\theta)$ 的 v 次微商. 更确切些, 对任一 $v\in \overset{\circ}{H}$, 常有

$$(u(\theta), \overline{v(\theta)}) = \sum_{\nu=0}^{\infty} c_\nu \overline{D^{(\nu)}(v(\psi))},$$

此处 $D^{(\nu)}(v(\psi)) = \frac{1}{\nu!}\left[\frac{d^\nu v(\psi+\arcsin x)}{dx^\nu}\right]_{x=0}$, 此级数是收敛的.

定理 2 S 类中仅有 ψ 为支点的广义函数可以表示为

$$u(\theta) = \sum_{\nu=0}^{l} c'_\nu \overline{\delta_\psi^{(\nu)}(\theta)},$$

此处 l 是一整正数. 更确切些：对任一 $v \in \overset{\circ}{S}$, 则有

$$(u(\theta), \overline{v(\theta)}) = \sum_{\nu=0}^{l} c_\nu \overline{v^{(\nu)}(\psi)}.$$

二定理之证明

并不失其普遍性, 可以假定 $\psi = 0$ 且把我们的区间移到 $-\pi \leqslant \theta \leqslant \pi$.

对任一已与的 $\varepsilon > 0$, 当 $r \to 1$ 时函数 $u(r, \theta)$ 在 $|\theta| \geqslant \varepsilon$ 中一致趋近于 0. 故

$$\lim_{r \to 1} \int_{-\pi}^{\pi} u(r, \theta)\overline{v(\theta)}d\theta = \lim_{r \to 1} \int_{-\varepsilon}^{\varepsilon} u(r, \theta)\overline{v(\theta)}d\theta.$$

当 ε 充分小时, 我们在 $|\theta| \leqslant \varepsilon$ 中把函数 $v(\theta)$ 展开为

$$v(\theta) = \sum_{v=0}^{l} D^{(\nu)}(v(0)) \sin^\nu \theta + R(\theta),$$

此处

$$R(\theta) = O(\varepsilon^{l+1}), D^{(\nu)}(v(0)) = \frac{1}{\nu!}\left(\frac{d^\nu v(\sin^{-1} x)}{dx^\nu}\right)_{x=0}$$

在定理 1 的假定下, 此级数可以展到无穷, 且当 $|\theta| \leqslant \varepsilon$ 时此级数一致收敛.

由于

$$\begin{aligned} c_\nu &= \lim_{r \to 1} \int_{-\varepsilon}^{\varepsilon} (\sin\theta)^\nu u(r, \theta)d\theta \\ &= \lim_{r \to 1} \int_{-\pi}^{\pi} (\sin\theta)^\nu u(r, \theta)d\theta \\ &= \lim_{r \to 1} \int_{-\pi}^{\pi} \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^\nu u(r, \theta)d\theta \\ &= \lim_{r \to 1} \frac{1}{(2i)^\nu} \sum_{t=0}^{\nu} \begin{pmatrix} \nu \\ t \end{pmatrix} \int_{-\pi}^{\pi} e^{it\theta} \cdot e^{-i(\nu-t)\theta} u(r, \theta)d\theta \\ &= \frac{1}{(2i)^\nu} \sum_{t=0}^{\nu} \begin{pmatrix} \nu \\ t \end{pmatrix} a_{\nu-2t}, \end{aligned}$$

故得定理 1(并且我们也给与了 c_ν 的表达式).

在证定理 2 时, 我们命

$$R^*(\theta) = \begin{cases} R(\theta), & 当|\theta| \leqslant \varepsilon, \\ 0, & 其他点. \end{cases}$$

则

$$\int_{-\varepsilon}^{\varepsilon} R(\theta)u(r,\theta)d\theta = \int_{-\pi}^{\pi} R^*(\theta)u(r,\theta)d\theta$$
$$= \int_{-\pi}^{\pi} R^*(\theta)(u(r,\theta)-a_0)d\theta + a_0\int_{-\varepsilon}^{\varepsilon} R(\theta)d\theta.$$

最后一项显然随 ε 而趋于 0. 又命 $w(r,\theta)$ 表一连续函数, 其 l 次微分等于 $u(r,\theta)-a_0$. 如此则

$$\lim_{r\to 1}\left|\int_{-\pi}^{\pi} R^*(\theta)(u(r,\theta)-a_0)d\theta\right| = \lim_{r\to 1}\left|\int_{-\pi}^{\pi} R^{*(l)}(\theta)w(r,\theta)d\theta\right|$$
$$\leqslant \int_{-\pi}^{\pi} |R^{*(l)}(\theta)w(\theta)|d\theta = O(\varepsilon).$$

故得定理 2.

§ 7. 其他类的广义函数

定义 命 ρ 表一正数, 如果

$$\overline{\lim_{n\to\infty}}\frac{\log|a_n|}{n\log n} < \frac{1}{\rho} \quad \overline{\lim_{n\to\infty}}\frac{\log|a_{-n}|}{n\log n} < \frac{1}{\rho},$$

则 $u(\theta)$ 称为 J_ρ 类的广义函数或 (J_ρ-广义函数).

显然, J_ρ 广义函数仍成一线性集, 且 J_ρ 广义函数的微分仍为 J_ρ 广义函数. 但是两个 J_ρ 广义函数的卷积不一定是 J_ρ 广义函数.

定理 1 类 J_ρ 的对偶类 $\mathring{J}_\rho$ 是由以下的广义函数所组成的:

$$\underline{\lim_{n\to\infty}}\frac{\log 1/|b_n|}{n\log n} \geqslant \frac{1}{\rho}, \quad \underline{\lim_{n\to\infty}}\frac{\log 1/|b_{-n}|}{n\log n} \geqslant \frac{1}{\rho}.$$

证 1) 由 J_ρ 的定义, 故对于一 $\delta(>0)$ 使当 $n \geqslant n_0(\delta)$ 时,

$$\frac{\log|a_n|}{n\log n} < \frac{1}{\rho} - \delta, \quad 即 |a_n| < n^{(\frac{1}{\rho}-\delta)n}.$$

另一方面, 由 $\mathring{J}_\rho$ 的定义, n 充分大时,

$$|b_n| < n^{-(\frac{1}{\rho}-\frac{1}{2}\delta)n},$$

故可知 $\sum a_n b_n$ 是收敛的.

2) 假定

$$\overline{\lim_{n\to\infty}}\frac{\log|a_n|}{n\log n} \geqslant \frac{1}{\rho},$$

则有一数列 n_ν 使

$$\lim_{\nu\to\infty}\frac{\log|a_{n_\nu}|}{n_\nu\log n_\nu}=\frac{1}{\sigma}\geqslant\frac{1}{\rho}.$$

取 $b_{n_\nu}=\dfrac{1}{a_{n_\nu}}$, 其他的 $b_n=0$, 则所得出的 $\nu(\theta)\in\mathring{J}_\rho$ 且 $\sum\limits_n a_nb_n$ 发散.

3) 假定

$$\varlimsup_{n\to\infty}\frac{\log\dfrac{1}{|b_n|}}{n\log n}<\frac{1}{\rho},$$

则有一数列 n_ν 使

$$\lim\frac{\log\dfrac{1}{|b_{n_\nu}|}}{n_\nu\log n_\nu}=\frac{1}{\tau}<\frac{1}{\rho}.$$

取 $a_{n_\nu}=\dfrac{1}{b_{n_\nu}}$, 其他的 $a_n=0$, 则得一 $u(\theta)\in J_\rho$ 且 $\sum\limits_n a_nb_n$ 发散.

对应于 J_ρ 类的一个广义函数 $u(\theta)$, 我们引进一个函数

$$u_\rho(r,\theta)=\sum_{m=0}^{\infty}p_mr^m\sum_{|n|\leqslant m}a_ne^{in\theta},$$

此处

$$p_m=\frac{1}{(m!)^{1/\rho}}.$$

由于有一 $\delta>0$ 使

$$\left|p_m\sum_{|n|\leqslant m}a_ne^{in\theta}\right|\leqslant\frac{\sum\limits_{|n|\leqslant m}|a_n|}{(m!)^{1/\rho}}=O\left(\frac{m\cdot m^{(\frac{1}{\rho}-\delta)m}}{m^{\frac{1}{\rho}m}}\right)=O(m^{-\frac{1}{2}\delta m}),$$

故 $u_\rho(r,\theta)$ 是一在全平面对任一紧致集绝对并一致收敛的级数.

定理 2 对任一 $u\in J_\rho$ 及任一 $v\in\mathring{J}_\rho$, 则

$$(u,\bar v)=\lim_{r\to\infty}\frac{1}{2\pi}\int_0^{2\pi}u_\rho(r,\theta)\overline{v(\theta)}d\theta/J_\rho(r),$$

此处

$$J_\rho(r)=\sum_{m=1}^{\infty}\frac{r^m}{(m!)^{1/\rho}}.$$

证 易知

$$\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{v(\theta)}d\theta=\sum_l\sum_{m=0}^{\infty}p_mr^m\sum_{|n|\leqslant m}a_n\bar b_l\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-l)\theta}d\theta$$

$$= \sum_{m=0}^{\infty} p_m r^m \sum_{|n| \leqslant m} a_n \bar{b}_n.$$

此定理可由广义 Borel 求和法的定理得之 (即附录 1 的定理 3).

同样我们可以定义致零集, 即为开区间 $a < \theta < b$ 使在其所有的子闭区间中皆一致地有

$$\lim_{r \to \infty} \frac{u_\rho(r, \theta)}{J_\rho(r)} = 0$$

的总集合, 同法可以处理仅有一个支点的广义函数.

注意, 由整函数论的知识 $\mathring{J}_\rho$ 就是所有的阶 $\leqslant \rho$ 的调和整函数的集合, 而 J_ρ 中所施行的求和法就是广义 Borel 求和法.

显然两个 J_ρ 广义函数的卷积不一定是一个 J_ρ 广义函数. 但是如果考虑适合于

$$\log |a_n| = O(|n| \log |n|)$$

的广义函数所成的集 J, 则 J 广义函数有以下的三性质: 线性集; 对微分自封; 卷积仍为 J-广义函数. 而 $\mathring{J}$ 则包有所有的零阶的整函数.

§ 8. (继　　续)

我们还可以更广些　命

$$Q(r) = \sum_{n=0}^{\infty} q_n r^n, \quad q_n \geqslant 0$$

表一幂级数对所有的 r 都收敛.

以 I_Q 表适合以下条件的广义函数的集合

$$\overline{\lim_{n \to \infty}} \frac{\log(|a_n| q_n)}{n \log n} < 0, \quad \overline{\lim_{n \to \infty}} \frac{\log(|a_{-n}| q_n)}{n \log n} < 0.$$

同法可证

定理 1　类 I_Q 的对偶类 $\mathring{I}_Q$ 乃由适合

$$\underline{\lim_{n \to \infty}} \frac{\log \frac{q_n}{|b_n|}}{n \log n} \geqslant 0, \quad \underline{\lim_{n \to \infty}} \frac{\log \frac{q_n}{|b_{-n}|}}{n \log n} \geqslant 0$$

的广义函数所组成.

又易证, 对 $u \in I_Q, v \in \mathring{I}_Q$ 常有

$$(u, \bar{v}) = \lim_{r \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \overline{v(\theta)} u_Q(r, \theta) d\theta / Q(r),$$

此处

$$u_Q(r,\theta)=\sum_{m=0}^{\infty}q_m r^m\sum_{|n|\leqslant m}a_n e^{in\theta}.$$

随便你给了怎样的广义函数, 我总可以选得合适的 q_n 使

$$\overline{\lim_{n\to\infty}}\frac{\log(|a_n|q_n)}{n\log n}<0,\ \overline{\lim_{n\to\infty}}\frac{\log(|a_{-n}|)q_n}{n\log n}<0,$$

换一句话说, 任何复杂的广义函数, 我们都有办法处理的.

§ 9. 极　　限

我们对任一非普通函数之广义函数类 T, 定义一极限如下: 设 $u_\nu(\theta)(\nu=1,2,\cdots)$ 是一串 T 的广义函数, $u(\theta)\in T$ 称为 $u_\nu(\theta)$ 的极限, 若对 T 的对偶类 $\overset{\circ}{T}$ 的任一函数 $v(\theta)$, 恒有

$$\lim_{\nu\to\infty}(u_\nu,\bar{v})=(u,\bar{v}).$$

至于 T 的对偶类 $\overset{\circ}{T}$ 的广义函数, 通常是普通意义下的函数. 我们在不同的情况下, 分别给以普通函数的收敛作为其极限:

(i) 设 $v_\nu(r,\theta)(\nu=1,2,\cdots)$ 是一个函数贯, 在一个包含闭单位圆的公共域内皆是调和的, 并且在此公共域内任一紧致子集一致收敛, 则显然的其极限函数 $v(r,\theta)$ 在公共域中也是调和的. 我们称 $v_\nu(\theta)$ 为 $v_\nu(\theta)$ 的极限, 或以 $v_\nu(\theta)\to v(\theta)(\overset{\circ}{H})$ 表之.

对任一 $u(\theta)\in H$, 我们取正数 δ 充分的小, 使 $v_\nu(r,\theta)$ 定义的公共域包含以 $r'=1+\frac{1}{2}\delta$ 为半径之闭圆. 由 §4 知

$$(u(\theta),v_\nu(\theta))=\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{v_\nu(r',\theta)}d\theta,\quad r=\frac{1}{1+\frac{1}{2}\delta}.$$

由假设 $v_\nu(r',\theta)$ 在半径为 r' 的闭圆一致收敛, 故有

$$\lim_{\nu\to\infty}(u,\bar{v}_\nu)=\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\overline{v_\nu(r',\theta)}d\theta=(u,\bar{v}).$$

(ii) 设 $v_\nu(r,\theta)(\nu=1,2,\cdots)$ 是一个函数贯, 在单位圆内调和, 在圆周上有无穷次微分, 并且 $\dfrac{\partial^p v_\nu(r,\theta)}{\partial\theta^p}$(对任意之非负整数 p) 在闭单位圆一致收敛, 则其极限函数 $v(r,\theta)$ 不难证明之也是在单位圆内调和, 在圆周上有无穷次微分. 我们称 $v_\nu(\theta)\to v(\theta)(\overset{\circ}{S})$.

任与 S 的函数 $u(\theta)=\sum a_n e^{in\theta}$. 设 $v_\nu(\theta)=\sum b_n^{(\nu)}e^{in\theta}$ 及 $v(\theta)=\sum b_n e^{in\theta}$. 由于 $v_\nu(\theta)$ 在圆周一致收敛, 故有

$$\lim_{\nu\to\infty} b_0^{(\nu)}=\lim_{\nu\to\infty}\frac{1}{2\pi}\int_0^{2\pi} v_\nu(\theta)d\theta=\frac{1}{2\pi}\int_0^{2\pi} v(\theta)d\theta=b_0.$$

由 §5 知, 不妨设 $u(\theta)$ 是连续函数 $w(\theta)$ 的 p 次微商. 我们有

$$(u,\bar{v}_\nu)=a_0 b_0^{(\nu)}+\frac{(-1)^p}{2\pi}\int_0^{2\pi} w(\theta)\overline{v_\nu^{(p)}(\theta)}d\theta.$$

因为假设知 $v_\nu^{(p)}(\theta)$ 在圆周上也是一致收敛为 $v^{(p)}(\theta)$, 故有

$$\lim_{\nu\to\infty}(u,\bar{v}_\nu)=a_0 b_0+\frac{(-1)^p}{2\pi}\int_0^{2\pi} w(\theta)\lim_{\nu\to\infty}\overline{v_\nu^{(p)}(\theta)}d\theta=(u,\bar{v}).$$

这是说, 如果 $v_\nu(\theta)\to v(\theta)(\mathring{S})$, 则恒有 ①

$$\lim_{\nu\to\infty}(u,\bar{v}_\nu)=(u,\bar{v}).$$

(iii) 设 $v_\nu(r,\theta)(\nu=1,2,\cdots)$ 是在整个平面调和的函数, 并在平面中任一紧致集一致收敛为一函数 $v(r,\theta)$, 则我们定义 $v_\nu(\theta)\to v(\theta)(\mathring{I})$. 从 (i) 的证明中知道必然有

$$\lim_{\nu\to\infty}(u,\bar{v}_\nu)=(u,\bar{v}).$$

(iv) 设 $v_\nu(r,\theta)(\nu=1,2,\cdots)$ 是 0 阶整调和函数, 在平面任一紧致集一致收敛为一函数 $v(r,\theta)$, 设亦为零阶的, 则定义 $v_\nu(\theta)\to v(\theta)(\mathring{J})$. 同样有

$$\lim_{\nu\to\infty}(u,\bar{v}_\nu)=(u,\bar{v}).$$

由以上可知, 对应于任一函数类 T 的对偶类 $\mathring{T}$, 我们总可以用相仿的方法适当的定义 $\mathring{T}$ 中的极限概念, 使得下面的关系成立:

$$\lim_{\nu\to\infty}(u,\bar{v}_\nu)=(u,\bar{v}).$$

这里不一一列举, 读者可试作之.

---

① 不难证明, 这里定义的 $\mathring{S}$ 类的极限是与 Schwartz 定义的极限是一样的. 但值得注意的, 这里定义 Schwartz 广义函数类 S 时, 并未有假定此等泛函之连续性, 而这里证明了连续性是必然包括的, 不必在定义中假定.

§ 10. 附　记

1) 从复变数函数论来看, 我们所引进的这些类是很自然的, 因为

类	类中函数的说明
$\overset{\circ}{K}$	有限和 (调和多项式)
$\overset{\circ}{G}_p$	0 阶 p 型的调和整函数
$\overset{\circ}{J}$	0 阶的调和整函数
$\overset{\circ}{I}$	所有的调和整函数
$\overset{\circ}{H}$	调和函数的有则域, 以闭单位圆为其内点者

并有关系

$$\overset{\circ}{K} \subset \overset{\circ}{J} \subset \overset{\circ}{I} \subset \overset{\circ}{H} \subset H \subset I \subset J \subset K.$$

利用整函数的阶在 $\overset{\circ}{J}$ 与 $\overset{\circ}{I}$ 之间, 我们还可以插入 $\overset{\circ}{J}_\rho$.

在 $\overset{\circ}{H}$ 与 H 之间还可以利用调和函数的边界性质插入其他类, 如 §3 例三所列: $\overset{\circ}{H} \subset \overset{\circ}{S} \subset L^{p'} \subset L^2 \subset L^p \subset S \subset H$.

另一方面, 从发散级数的求和的理论, 我们的讨论也是有它的系统性的意义的.

2) 由 §3 的定理 1 与 4 可知, 从无穷大之阶来分类也是极自然的, 而主要的类是

类	$\log\lvert a_n\rvert$ 的阶
S	$O(\log n)$
H	$o(n)$
I	$O(n)$
J	$O(n\log n)$
G_p	$o(n^p),\quad p>1.$

3) 由保角映象的原理可知, 我们所讨论的类 H, 实质上并不限于单位圆. 例如, 我们可以一样地研究实数轴与上半平面.

另一方面, 由于 Fourier 级数与 Fourier 积分的相似性质, 我们可以直接处理实数轴. 先引进形式 Fourier 积分

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(t)e^{itx}dt.$$

如果对任一 $\varepsilon > 0, \log|a(t)| = o(1)$, 则 $u(x)$ 称为属于 H 类. 如果有某一 $p(>0)$ 使 $a(t) = O(|t|^p)$, 则 $u(x)$ 称为属于 S 类. 其对应的调和函数是

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(t) e^{itx-|t|y} dt, \quad y > 0.$$

4) 我们也可以用任一椭圆型的偏数分方程来代替 Laplace 方程. 如此对应于域内的一个解, 我们在边界上可以定义一个广义函数. 当然, 还可以推得广些, 如我所研究过典型域的复变数函数和调和函数等等 (华罗庚[4], 华罗庚与陆启铿[5]).

5) 更一般些, 我们可以推广 §3 的定理 1 及 4 入手.

命 R_n 表 n 维实欧氏空间, $t = (t_1, \cdots, t_n)$ 表其中的点. 又命 $\tau = \sqrt{t_1^2 + \cdots + t_n^2}$. 与定理 1 相仿我们有以下的结果.

命 $\varphi(\tau)$ 表一单变数 $\tau(\geqslant 0)$ 的正递增函数, 且以任一 $\delta > 0$, 积分

$$\int_0^{\infty} \varphi(\tau)^{-\delta} \tau^{n-1} d\tau \tag{1}$$

常收敛.

命 A 代表适合以下条件的函数 $a(t)$:

(i) 在任一有限区间中 $a(t)$ 是平方可积及 (ii) 除一测度为零的集合外, 当 τ 充分大时

$$\log|a(\tau)| = o(\log \varphi(\tau)). \tag{2}$$

又命 B 代表适合以下条件的函数 $b(t)$: (i) 在任一有限区间中 $b(t)$ 是平方可积及 (ii) 除一测度为零的集合外, 当 τ 充分大时

$$\log \varphi(\tau) = O\left(\log \frac{1}{|b(\tau)|}\right). \tag{3}$$

如此 A 与 B 之间有以下的三个性质:

(i) 若 $a(t) \in A, b(t) \in B$, 则

$$\int_{R_n} a(t)\overline{b(t)} dt < \infty, \tag{4}$$

此处 $dt = dt_1 \cdots dt_n$;

(ii) 若对 B 中的任一 $b(t)$, (4) 常收敛, 则 $a(t) \in A$;

(iii) 若对 A 中的任一 $a(t)$, (4) 常收敛, 则 $b(t) \in B$.

证明从略, 且有与定理 4 相仿的结果. 命 $\overset{\circ}{K}$ 表

$$v(x) = \frac{1}{(\sqrt{2\pi})^n} \int_{R_n} b(t) e^{-itx'} dt \tag{5}$$

所成的类, 此处 $tx' = t_1x_1 + \cdots + t_nx_n$. 此积分是绝对收敛的.

积分

$$\int_{R_n} a(t)\overline{b(t)}dt$$

可以看作为 $a(t)(\in A)$ 的一个线性泛函. 因此而定义了一个广义函数 $u(x)$, 且

$$(u(x), \overline{v(x)}) = \int_{R_n} a(t)\overline{b(t)}dt.$$

此 $u(x)$ 可以形式 Fourier 积分

$$u(x) = \frac{1}{(\sqrt{2\pi})^n}\int_{R_n} a(t)e^{-itx'}dt$$

表示, 此 $u(x)$ 可由以下的方法实现出来:

i) 若对任一 ε,

$$a(t) = O(e^{\varepsilon|t|}),$$

则用

$$u(x, y) = \frac{1}{(\sqrt{2\pi})^n}\int_{R_n} a(t)e^{-itx'-|t|y'}dt,$$

此处 $|t|y' = |t_1|y_1 + \cdots + |t_n|y_n$. 而

$$(u(x), \overline{v(x)}) = \lim_{y\to 0}\int_{R_n} u(x, y)\overline{v(x)}dx.$$

ii) 其他的情况可引进

$$u(x, r) = \int_0^\infty \frac{r^\nu d\nu}{\varphi(\nu)}\int_{\tau\leqslant\nu} a(t)e^{-itx'}dt \Big/ \int_0^\infty \frac{r^\nu}{\varphi(\nu)}d\nu,$$

而

$$(u(x), \overline{v(x)}) = \lim_{r\to\infty}\int_{R_n} u(x, r)\overline{v(x)}dx.$$

此法比 Schwartz 的方法有显著的优点.

附录 I 求 和 法

定理 1 命 $q_\nu(r)$ 为一函数列, 在一以 r_0 为右端的区间中定义, 且有以下的性质:

(i) $q_\nu(r) \geqslant 0$;

(ii) $\sum_{\nu=0}^{\infty} q_\nu(r) = 1, r < r_0$,

及对任一 ν 且

(iii) $\lim\limits_{r\to r_0} q_\nu(r) = 0$.

如此, 则由 $s_n \to s$ 可得

$$\lim_{r\to r_0}\sum_{\nu=0}^{\infty} q_\nu(r)s_\nu = s.$$

证　并不失去普遍性可以假定 $s = 0$. 对任与 $\varepsilon > 0$, 我们有 M 使当 $v \geqslant M$ 时 $|s_\nu| < \varepsilon$. 并可假定对所有的 $\nu, |s_\nu| \leqslant B$. 则

$$\left|\sum_{\nu=0}^{\infty} q_\nu(r)s_\nu\right| \leqslant B\sum_{\nu=0}^{M} q_\nu(r) + \varepsilon \sum_{\nu=M+1}^{\infty} q_\nu(r) \leqslant B\sum_{\nu=0}^{M} q_\nu(r) + \varepsilon.$$

当 $r \to r_0$ 时, 右边趋于 ε. 故得定理.

现在取两个重要的特例:

1) 取 $r_0 = 1, q_\nu(r) = (1-r)r^\nu$. 条件 (i), (ii), (iii) 都合, 故当 $r \to 1$ 时

$$\sum_{\nu=0}^{\infty}(1-r)r^\nu s_\nu = \sum_{\nu=0}^{\infty}(s_\nu - s_{\nu-1})r^\nu \to \lim_{\nu\to\infty} s_\nu.$$

即

定理 2 (Abelien)　若

$$\sum_{\nu=0}^{\infty} a_\nu$$

收敛于 s, 则当 $r \to 1$ 时

$$\sum_{\nu=0}^{\infty} a_\nu r^\nu$$

也趋于 s.

2) 假定

$$\sum_{\nu=0}^{\infty} p_\nu r^\nu, \quad p_\nu \geqslant 0$$

是一到处收敛的幂级数. 命

$$q_\nu(r) = \frac{p_\nu r^\nu}{\sum\limits_{\nu=0}^{\infty} p_\nu r^\nu}.$$

则 (i), (ii) 显然适合, 由于

$$\lim_{r\to\infty} q_\nu = \lim_{r\to\infty}\frac{\nu p_\nu r^{\nu-1}}{\sum\limits_{\nu=1}^{\infty}\nu p_\nu r^{\nu-1}} = \cdots = 0,$$

故得

定理 3 (广义 Borel 求法定理) 若

$$\sum_{\nu=0}^{\infty} p_\nu r^\nu, \quad p_\nu \geqslant 0$$

是一到处收敛的幂级数. 则当 $s_\nu \to s$ 时

$$\lim_{r\to\infty} \frac{\sum_{\nu=0}^{\infty} p_\nu r^\nu s_\nu}{\sum_{\nu=0}^{\infty} p_\nu r^\nu} = s.$$

定理 1 还有其相似类的积分定理

定理 4 命 $q(r,\theta)$ 是一函数在 $0 \leqslant \theta \leqslant 2\pi$ 及 $0 < r < 1$ 中定义且有次之性质:

(i) $q(r,\theta) \geqslant 0$.

(ii) $\int_0^{2\pi} q(r,\theta)d\theta = 1$

及对任一 $\varepsilon > 0$

$$\lim_{r\to 1} \int_{|\theta|\geqslant\varepsilon} q(r,\theta)d\theta = 0,$$

如此, 若当 $\theta \to \pm 0$ 时 $f(\theta) \to s$, 则

$$\lim_{r\to 1} \int_0^{2\pi} q(r,\theta)f(\theta)d\theta = s$$

(此处假定 $f(\theta)$ 是囿函数).

附录 II 调 和 函 数

定义 一个函数 $u(r,\theta)$ 有二级连续偏导数, 且对 θ 以 2π 为周期, 并当 $0 \leqslant r < r_0$ 时, 这函数适合于 Laplace 微分方程

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u(r,\theta)\right) + \frac{\partial^2 u(r,\theta)}{\partial\theta^2} = 0, \tag{1}$$

则此函数称为圆 $(0 \leqslant \theta \leqslant 2\pi, 0 \leqslant r < r_0)$ 内的调和函数.

研究积分

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)e^{-in\theta}d\theta = y(r). \tag{2}$$

积分号下求微分, 即得

$$\frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u(r,\theta)\right)e^{-in\theta}d\theta = r\frac{d}{dr}\left(r\frac{d}{dr}y(r)\right).$$

另一方面, 部分积分两次, 可知, 当 $n\neq 0$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2 u(r,\theta)}{\partial\theta^2}e^{-in\theta} = -n^2\int_0^{2\pi}u(r,\theta)e^{-in\theta}d\theta = -n^2y(r).$$

由 (1) 可知

$$r\frac{d}{dr}\left(r\frac{d}{dr}y(r)\right) = -n^2y(r). \tag{3}$$

显然, r^n 及 r^{-n} 都是 (3) 式的解, 故 (3) 的一般解的形式是 $C_1r^n + C_2r^{-n}$, 因为 $\lim\limits_{r\to 0}y(r)$ 是有限的, 故

$$y(r) = a_nr^{|n|}.$$

当 $n=0$ 时, 同法可以得出同样的结果.

故, 当 $0\leqslant r\epsilon r_0$ 时, 连续函数 $u(r,\theta)$ 有 Fourier 级数展开式

$$u(r,\theta) = \sum_{n=-\infty}^{\infty}a_ne^{in\theta}r^{|n|}. \tag{4}$$

极易证明, 对任一 ε, 级数 (4) 在圆 $(0\leqslant\theta\leqslant 2\pi, 0\leqslant r\leqslant r_0-\varepsilon)$ 内一致收敛. 由 (4) 可以对 θ 及 r 无穷次微分. 级数 (4) 显然适合方程式 (1).

对已知的 a_n, 今往求最大的 r_0 使 (4) 当 $r<r_0$ 时收敛. 若 (4) 收敛, 显然有

$$|a_ne^{in\theta} + a_{-n}e^{-in\theta}|r^{|n|} = O(1),$$

即

$$\varliminf_{n\to\infty}|a_ne^{in\theta} + a_{-n}e^{-in\theta}|^{-\frac{1}{n}} \geqslant r_0. \tag{5}$$

另一方面, 若 (5) 成立, 则对任一 ε, 当 n 充分大时常有

$$|a_ne^{in\theta} + a_{-n}e^{-in\theta}| \leqslant \frac{1}{(r_0-\varepsilon)^n}.$$

故级数

$$a_0 + \sum_{n=1}^{\infty}(a_ne^{in\theta} + a_{-n}e^{-in\theta})r^n$$

当 $r<r_0-\varepsilon$ 时收敛.

故 (4) 代表单位圆内的调和函数的必要且充分的条件是, 对所有的 θ

$$\overline{\lim_{n\to\infty}}|a_n e^{in\theta}+a_{-n}e^{-in\theta}|^{\frac{1}{n}}\leqslant 1.$$

这一条件可以以下的简洁的条件

$$\max(\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}},\overline{\lim_{n\to\infty}}|a_{-n}|^{\frac{1}{n}})\leqslant 1$$

代替它, 其原因是基于以下的

引理 我们有

$$\max_{0\leqslant\theta\leqslant 2\pi}\overline{\lim_{n\to\infty}}|a_n e^{in\theta}+b_n|^{\frac{1}{n}}=\max(\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}},\overline{\lim_{n\to\infty}}|b_n|^{\frac{1}{n}}).$$

证 显然有

$$\begin{aligned}\overline{\lim_{n\to\infty}}|a_n e^{in\theta}+b_n|^{\frac{1}{n}}&\leqslant\overline{\lim_{n\to\infty}}(2\max(|a_n|,|b_n|))^{\frac{1}{n}}\\&=\max(\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}},\overline{\lim_{n\to\infty}}|b_n|^{\frac{1}{n}}).\end{aligned}$$

命

$$\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}}=\alpha,\quad\overline{\lim_{n\to\infty}}|b_n|^{\frac{1}{n}}=\beta,$$

并不失却普遍性可以假定 $\beta\geqslant\alpha$, 若有一数列 n_ν 使

$$\lim_{\nu\to\infty}|b_{n_\nu}|^{\frac{1}{n_\nu}}=\beta,\quad\overline{\lim_{\nu\to\infty}}|a_{n_\nu}|^{\frac{1}{n_\nu}}=\alpha'<\beta,$$

则

$$\begin{aligned}\overline{\lim_{\nu\to\infty}}|a_{n_\nu}e^{in\theta}+b_{n_\nu}|^{\frac{1}{n_\nu}}&=\overline{\lim_{\nu\to\infty}}\left|1+\frac{a_{n_\nu}}{b_{n_\nu}}e^{in_\nu\theta}\right|^{\frac{1}{n_\nu}}|b_{n_\nu}|^{\frac{1}{n_\nu}}\\&\geqslant\overline{\lim_{\nu\to\infty}}\left|\frac{1}{2}\left(1-\frac{\alpha'}{\beta}\right)\right|^{\frac{1}{n_\nu}}|b_{n_\nu}|^{\frac{1}{n_\nu}}=\beta.\end{aligned}$$

所以现在要研究的情况是: 有一数列 n_ν 使

$$\lim_{\nu\to\infty}|a_{n_\nu}|^{\frac{1}{n_\nu}}=\lim_{\nu\to\infty}|b_{n_\nu}|^{\frac{1}{n_\nu}}=\beta.$$

命

$$\frac{a_{n_\nu}}{b_{n_\nu}}=\rho_{n_\nu}e^{i\tau_\nu},\quad 0\leqslant\tau_\nu\leqslant 2\pi.$$

若集合 τ_ν 有一极限点 $\tau\neq\pi$, 则取 $\theta=0$, 我们有一数列 n_ν 使

$$\overline{\lim_{\nu\to\infty}}|a_{n_\nu}+b_{n_\nu}|^{\frac{1}{n_\nu}}\geqslant\overline{\lim_{\nu\to\infty}}\left(\frac{1}{2}|1+e^{i\tau}|\right)^{\frac{1}{n_\nu}}|b_{n_\nu}|^{\frac{1}{n_\nu}}=\beta.$$

所以最后需要研究的是

$$\lim_{\nu\to\infty}\tau_\nu=\pi$$

的情况. 命 l_ν 是 α 的最高方次之能整除 n_ν 者. 若有一整数 l 在 $\{l_\nu\}$ 中出现无穷次, 则取 $\theta=\frac{\pi}{2^l}$ 即能证所欲证. 今设

$$l'_1<l'_2<l'_3<\cdots$$

为 $\{l_\nu\}$ 中的不同的 l. 取

$$\theta=\pi(\alpha^{-l'_1}+\alpha^{-l'_2}+\alpha^{-l'_3}+\cdots).$$

这是收敛的级数. 命 n'_ν 恰为 $\alpha^{l'_{2\nu+1}}$ 所整除的整数, 则仍得

$$\overline{\lim_{\nu\to\infty}}|a_{n'_\nu}e^{in'_\nu\theta}+b_{n'_\nu}|\geqslant\beta.$$

故引已证明

定理 1 当 $r\leqslant r_0(4)$ 表一调和函数的必要且充分条件是

$$\max(\overline{\lim_{n\to\infty}}|a_n|^{\frac{1}{n}},\quad\overline{\lim_{n\to\infty}}|a_{-n}|^{\frac{1}{n}})\leqslant\frac{1}{r_0}.$$

命 $u(\theta)$ 表一以 2π 为周期的连续 (或可积) 函数, 且

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}u(\theta)e^{-in\theta}d\theta \tag{6}$$

表 $u(\theta)$ 的 Fourier 系数. 作

$$u(r,\theta)=\sum_{n=-\infty}^{\infty}a_ne^{in\theta}r^{|n|}. \tag{7}$$

由于 $a_n=O(1)$, 故此级数当 $r<1$ 时表一调和函数.

以 (6) 代入 (7), 可知

$$\begin{aligned}u(r,\theta)&=\sum_n e^{in\theta}r^{|n|}\frac{1}{2\pi}\int_0^{2\pi}u(\psi)e^{-in\psi}d\psi\\&=\frac{1}{2\pi}\int_0^{2\pi}u(\psi)\sum_n e^{in\theta}r^{|n|}e^{-in\psi}d\psi\\&=\frac{1}{2\pi}\int_0^{2\pi}u(\psi)P(r,\theta-\psi)d\psi\\&=\frac{1}{2\pi}\int_0^{2\pi}u(\theta-\psi)P(r,\psi)d\psi,\end{aligned}$$

此处

$$P(r,\theta-\psi)=\sum_n e^{in\theta}r^{|n|}e^{-in\psi}$$
$$=(1-r^2)/(1-2r\cos(\theta-\psi)+r^2).$$

由附录 I 定理 4 立得

$$\lim_{r\to 1}u(r,\theta)=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}u(\theta-\psi)P(r,\psi)d\psi=u(\theta).$$

参 考 文 献

[1] Гельфанд, И. М., Шилов Г Е. Преобразования фурве бЫстро растущих функций и вопросы единственности решения задачй. *УМН*, 8: 9(1953), 3-54.

[2] Schwartz L., La theoire des distributions, I, II, 1951, Paris.

[3] 冯康. 广义函数论. 数学进展, 1955, 1: 405-590.

[4] 华罗庚. On a system of partial differential equations. *Science Rec.(N.S.)*, 1957, 1: 369-371.

[5] 华罗庚, 陆启铿. Theory of Harmonic functions in classical domains. *Scientia Sinica*, 1959, 8: 1031-1094.

常系数二阶椭圆型偏微分方程组 Dirichlet 问题的唯一性定理

§ 1. 引　　言

命 A, B, C 代表三个两行两列的实数方阵, 矩阵形式的偏微分方程

$$A\frac{\partial^2}{\partial x^2}\begin{pmatrix} u \\ v \end{pmatrix} + 2B\frac{\partial^2}{\partial x \partial y}\begin{pmatrix} u \\ v \end{pmatrix} + C\frac{\partial^2}{\partial y^2}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{1.1}$$

实际上是两个自变数 x, y, 两个未知函数 u, v 的两个方程所成的方程组.

行列式

$$Q(\xi, \eta) = |A\xi^2 + 2B\xi\eta + C\eta^2|$$

定义为偏微分方程组 (1.1) 的特征四次型. И. Г. Петровский 定义: 如果特征四次型的根是两对复的, 则 (1.1) 称为椭圆型. 这样的定义不能保证 Dirichlet 问题的唯一性. 例如, 设有实数 β 及 γ 适合

$$|A + 2\beta B + \gamma C| = 0, \quad \beta^2 < \gamma, \tag{1.2}$$

则必有一矢量 $\begin{pmatrix} a \\ b \end{pmatrix}$ 使

$$(A + 2\beta B + \gamma C)\begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{1.3}$$

显然,

$$u = a(x^2 + 2\beta xy + \gamma y^2 - 1), \quad v = b(x^2 + 2\beta xy + \gamma y^2 - 1)$$

是 (1.1) 的解答, 它们在椭圆 $x^2 + 2\beta xy + \gamma y^2 = 1$ 上都等于 0. 为了使解唯一, 我们必须在Петровский条件外, 再加一些条件.

1951 年, М. И. Вишик[1] 引进了强椭圆组的概念: 对任一满秩方阵 $P, A_1 = PA, B_1 = PB, C_1 = PC$ 所对应的偏微分方程组实质上是 (1.1) 的两个方程的线性组合. 如果有一 P 使

$$\widetilde{A}_1 + 2\widetilde{B}_1 t + \widetilde{C}_1 t^2 > 0, \tag{A}$$

∗ 1963 年 10 月 10 日收到, 1964 年 11 月 11 日收到修改稿. 发表于《数学学报》, 1965, 15(2): 242–248. 作者: 华罗庚 (中国科学院数学研究所, 中国科学技术大学), 吴兹潜, 林伟 (中山大学).

则 (1.1) 定义为强椭圆组. 这儿 $\widetilde{H}=\frac{1}{2}(H+H'), H'$ 是 H 的转置方阵, 又"$S>0$"表示对称方阵 S 是定正的. Вищик 证明了强椭圆组的 Dirichlet 问题的解的唯一性.

1960 年, 丁夏畦 [2] 等五位同志得出以下的结果: 如果 (1.2) 无解, 简称为条件 (B), 则对任意有限区域的 Dirichlet 问题的解是唯一的; 并且其逆也真.

本文的目的在于证明条件 (A), (B) 的等价性. 为了读者方便起见, 我们并不引用以往文献上的知识; 而且还顺便证明了唯一性定理 (§5), 它似乎比原来的处理方法较简单些.

§ 2. 如果 (A) 适合, 则 (B) 也适合

如果式 (1.2) 有解, 则有式 (1.3). 因此, 对任一 $P, |P|\neq 0$, 令 $A_1=PA, B_1=PB, C_1=PC$, 总有

$$(A_1+2\beta B_1+\gamma C_1)\begin{pmatrix} a \\ b \end{pmatrix}=0,$$

从而

$$(a,b)(A_1+2\beta B_1+\gamma C_1)\begin{pmatrix} a \\ b \end{pmatrix}=0,$$

转置相加除 2, 即得

$$(a,b)(\widetilde{A}_1+2\beta\widetilde{B}_1+\gamma\widetilde{C}_1)\begin{pmatrix} a \\ b \end{pmatrix}=0.$$

当然 $\widetilde{A}_1+2\beta\widetilde{B}_1+\gamma\widetilde{C}_1$ 不可能定正. 即 (A) 不适合.

§ 3. 适合 (B) 的方程组的标准型

方程组 (1.1) 和方程组

$$A_1\frac{\partial^2}{\partial x^2}\begin{pmatrix} u \\ v \end{pmatrix}+2B_1\frac{\partial^2}{\partial x\partial y}\begin{pmatrix} u \\ v \end{pmatrix}+C_1\frac{\partial^2}{\partial y^2}\begin{pmatrix} u \\ v \end{pmatrix}=0 \qquad (1.1)'$$

的等价关系是由以下三种运算来定义的:

i) 方程间的线性组合: 即

$$A_1=PA,\quad B_1=PB,\quad C_1=PC,\quad |P|\neq 0;$$

ii) 未知函数的线性变换：即

$$A_1 = AQ, \quad B_1 = BQ, \quad C_1 = CQ, \quad |Q| \neq 0;$$

iii) 自变数的线性变换：即

$$A_1 = p^2 A + 2pqB + q^2 C,$$
$$B_1 = prA + (ps+qr)B + qsC, \quad \begin{vmatrix} p & q \\ r & s \end{vmatrix} \neq 0,$$
$$C_1 = r^2 A + 2rsB + s^2 C.$$

特征四次型经变换 i), ii) 后仅差一个常数因子. 经变换 iii) 变为

$$Q_1(\xi, \eta) = |A_1\xi^2 + 2B_1\xi\eta + C_1\eta^2| = Q(\xi', \eta'),$$

这儿 $\xi' = p\xi + r\eta, \eta' = q\xi + s\eta$. 命 $\dfrac{\xi_i}{\eta_i} = \tau_i, \dfrac{\xi_i'}{\eta_i'} = \tau_i'$ $(i = 1, 2, 3, 4)$ 分别为 $Q_1(\tau, 1) = 0$ 及 $Q(\tau', 1) = 0$ 的根, 则

$$\tau_i' = \frac{p\tau_i + r}{q\tau_i + s} \quad (i = 1, 2, 3, 4),$$

这是一次分式变换.

命 $\tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2$ 是特征四次型的四个复根, 过这四点可作一圆与实轴正交. 显然存在具有实系数的一次分式变换将这圆变为虚轴, 把 $\tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2$ 变为 $\pm ki(k > 0), \pm i$, 于是立得

$$Q(\xi, \eta) = |A\xi^2 + 2B\xi\eta + C\eta^2| = (\xi^2 + \eta^2)(\xi^2 + k^2\eta^2), \quad k > 0, \tag{1.4}$$

所以不妨假设 $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. 对矩阵 C 在实数范围内存在 P 使 $PCP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}$. 现在分三种情况来讨论:

1) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, C = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

由 (1.4) 比较系数得 $b_1 = b_4 = 0$, 即得

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_2 \\ b_3 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \tag{1.5}$$
$$\lambda + \mu - 4b_2b_3 = \lambda\mu + 1, \quad \lambda\mu > 0.$$

1.1) 若 $b_2b_3<0$, 左乘 $\begin{pmatrix}\frac{1}{b_2} & 0\\ 0 & -\frac{1}{b_2b_3}\end{pmatrix}$, 右乘 $\begin{pmatrix}b_2 & 0\\ 0 & 1\end{pmatrix}$, 得标准型

$$A=\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix},\quad B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad C=\begin{pmatrix}\lambda & 0\\ 0 & \mu_1\end{pmatrix},\tag{I$_0$}$$

$$\alpha\lambda+\mu_1+4=\lambda\mu_1+\alpha,\quad \alpha>0,\quad \lambda\mu_1>0,$$

这儿, $\alpha=-\dfrac{1}{b_2b_3},\mu_1=\alpha\mu$.

1.2) 若 $b_2b_3>0$, 左乘 $\begin{pmatrix}\frac{1}{b_2} & 0\\ 0 & \frac{1}{b_2b_3}\end{pmatrix}$, 右乘 $\begin{pmatrix}b_2 & 0\\ 0 & 1\end{pmatrix}$, 得

$$A=\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix},\quad B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad C=\begin{pmatrix}\lambda & 0\\ 0 & \mu_1\end{pmatrix},\tag{II$_0$}$$

$$\alpha\lambda+\mu_1-4=\lambda\mu_1+\alpha,\quad \alpha>0,\quad \lambda\mu_1>0.$$

1.3) 若 $b_2=0,b_3\neq 0$, 左乘 $\begin{pmatrix}1 & 0\\ 0 & \frac{1}{b_3}\end{pmatrix}$, 右乘 $\begin{pmatrix}1 & 0\\ 0 & b_3\end{pmatrix}$, 得

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad C=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}.$$

因 $\lambda+\mu=\lambda\mu+1,\lambda\mu>0$, 故得 $(\lambda-1)(\mu-1)=0$. 即得

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad C=\begin{pmatrix}1 & 0\\ 0 & \mu\end{pmatrix},\quad \mu>0.\tag{III$_0$}$$

(如果 $b_2\neq 0,b_3=0$, 则仍可化为标准型 (III$_0$)).

1.4) $b_2=b_3=0$. 此时 $B=0,\lambda=1$(或 $\mu=1$). 立得

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\quad C=\begin{pmatrix}1 & 0\\ 0 & \mu\end{pmatrix},\quad \mu>0.\tag{IV$_0$}$$

2) $A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},B=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix},C=\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix}$.

将 A,B,C 代入 (1.4), 比较 $\xi^3\eta$ 及 $\xi\eta^3$ 的系数, 得 $b_4=-b_1, b_2=0$; 再比较 $\xi^2\eta^2$ 及 η^4 的系数, 得 $-4b_1^2=(\lambda-1)^2$, 立得 $b_1=b_4=0, \lambda=1$, 即得

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\b_3&0\end{pmatrix},\quad C=\begin{pmatrix}1&0\\1&1\end{pmatrix}.$$

作自变数变换:

$$\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},$$

得

$$A=\begin{pmatrix}1&0\\a&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\b_3\cos 2\theta+\frac{1}{2}\sin 2\theta&0\end{pmatrix},\quad C=\begin{pmatrix}1&0\\c&1\end{pmatrix},$$

这儿 $a=b_3\sin 2\theta+\sin^2\theta, c=-b_3\sin 2\theta+\cos^2\theta$. 取 $\theta=\frac{1}{2}\arctan\frac{1}{2b_3}$, 则 $a=c$, 左乘 $\begin{pmatrix}1&0\\-a&1\end{pmatrix}$, 得

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\b&0\end{pmatrix},\quad C=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

它仍可化为标准型 (III_0) 或 (IV_0).

3) $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}, B=\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix}, C=\begin{pmatrix}\lambda&\mu\\-\mu&\lambda\end{pmatrix}$.

代入 (1.4), 比较系数. 若 $\mu\neq 0$, 得 $-4(b_1^2+b_2^2)=(\lambda-1)^2+\mu^2$, 这是不可能的. 若 $\mu=0$, 则得

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix},\quad C=\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix},$$

它归结到情况 1).

(I_0)—(IV_0) 是椭圆型方程组 (Петровский 定义下) 的标准型.

将标准型 (III_0), (IV_0) 代入式 (1.2), 易见条件 (B) 成立. 对于标准型 (I_0), (II_0), 条件 (B) 等价于 $\lambda>0, \mu_1>0$. 事实上, 如果 $\lambda>0, \mu_1>0$, 则当 $\gamma>\beta^2$ 时, 对 (I_0) 有

$$|A+2\beta B+\gamma C|=\begin{vmatrix}1+\gamma\lambda&2\beta\\-2\beta&\alpha+\gamma\mu_1\end{vmatrix}=(1+\gamma\lambda)(\alpha+\gamma\mu_1)+4\beta^2>0,$$

对于 (II_0) 有

$$|A+2\beta B+\gamma C|=\begin{vmatrix} 1+\gamma\lambda & 2\beta \\ 2\beta & \alpha+\gamma\mu_1 \end{vmatrix}=\alpha+(\alpha\lambda+\mu_1)\gamma+\lambda\mu_1\gamma^2-4\beta^2$$
$$>\alpha+(\alpha\lambda+\mu_1-4)\beta^2+\lambda\mu_1\gamma^2>0,$$

即条件 (B) 成立. 如果 $\lambda<0(\mu_1<0)$, 则取$\gamma=-\dfrac{1}{\lambda}\left(\gamma=-\dfrac{1}{\mu_1}\right),\beta=0$, 显然 $\beta^2<\gamma$, 但 $|A+2\beta B+\gamma C|=0$, 即 (B) 不成立. 故得下述定理:

定理 1 经过自变数的线性变换, 函数的线性变换及方程式的线性组合, 任何一个适合条件 (B) 的椭圆型方程组一定可变为下面四种标准型之一:

$$A=\begin{pmatrix}1&0\\0&\alpha\end{pmatrix},\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad C=\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix},\tag{I}$$
$$\alpha\lambda+\mu+4=\lambda\mu+\alpha,\quad \alpha>0,\quad \lambda>0,\quad \mu>0;$$
$$A=\begin{pmatrix}1&0\\0&\alpha\end{pmatrix},\quad B=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad C=\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix},\tag{II}$$
$$\alpha\lambda+\mu-4=\lambda\mu+\alpha,\quad \alpha>0,\quad \lambda>0,\quad \mu>0;$$
$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad C=\begin{pmatrix}1&0\\0&\mu\end{pmatrix},\quad \mu>0;\tag{III}$$
$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad C=\begin{pmatrix}1&0\\0&\mu\end{pmatrix},\quad \mu>0.\tag{IV}$$

§ 4. 如果 (B) 适合, 则 (A) 也适合

由定理 1 可知, 任何一个适合条件 (B) 的方程组一定等价于标准型 (I)—(IV), 显然这都适合于 $\widetilde{A}+2\widetilde{B}t+\widetilde{C}t^2>0$. 事实上, 由 (I)—(IV), 依次有

$$\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}+\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}t^2,\qquad \begin{matrix}\alpha\lambda+\mu+4=\lambda\mu+\alpha,\\ \alpha>0,\lambda>0,\mu>0,\end{matrix}$$

$$\begin{pmatrix}1&0\\0&\alpha\end{pmatrix}+\begin{pmatrix}0&2\\2&0\end{pmatrix}t+\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}t^2,\qquad \begin{matrix}\alpha\lambda+\mu-4=\lambda\mu+\alpha,\\ \alpha>0,\lambda>0,\mu>0,\end{matrix}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&1\\1&0\end{pmatrix}t+\begin{pmatrix}1&0\\0&\mu\end{pmatrix}t^2,\quad \mu>0,$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&\mu\end{pmatrix}t^2,\qquad \mu>0.$$

它们的主对角线上的元素均为正, 而且行列式亦为正, 因此得下述定理:

定理 2 条件 (A), (B) 互相等价.

§ 5. Dirichlet 问题的解的唯一性

命 C 是一逐段光滑的 Jordan 闭曲线, 包有一个域 D, 如果 $u|_c = 0, v|_c = 0$, 就有方程组的解恒等于 0, 则称方程组的 Dirichlet 问题的解有唯一性.

定理 3 椭圆型方程组 Dirichlet 问题的解的唯一性的必要充分条件是条件 (B), 也是条件 (A).

证 在 §1 中已经指明, 条件 (B) 不适合, 则解必非唯一. 现在只需证明方程组 (I)—(IV) 的 Dirichlet 问题的解是唯一的.

方程组 (III) 及 (IV) 可以分离为两个椭圆型方程来考虑, 显然它们的 Dirichlet 问题的解是唯一的; 我们只要考虑 (I) 及 (II) 的唯一性就够了.

1) 标准型 (I) 所对应的方程组是

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 v}{\partial x \partial y} + \lambda \dfrac{\partial^2 u}{\partial y^2} = 0, \\ \alpha \dfrac{\partial^2 v}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \mu \dfrac{\partial^2 v}{\partial y^2} = 0. \end{cases} \tag{I$'$}$$

从等式

$$\begin{aligned} &\frac{\partial}{\partial x}\left[u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\left(\alpha\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \\ &+ \frac{\partial}{\partial y}\left[u\left(\frac{\partial v}{\partial x} + \lambda\frac{\partial u}{\partial y}\right) + v\left(-\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y}\right)\right] \\ =&\left(\frac{\partial u}{\partial x}\right)^2 + \lambda\left(\frac{\partial u}{\partial y}\right)^2 + \alpha\left(\frac{\partial v}{\partial x}\right)^2 + \mu\left(\frac{\partial v}{\partial y}\right)^2 + u\left[\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y} + \lambda\frac{\partial^2 u}{\partial y^2}\right] \\ &+ v\left[\alpha\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \mu\frac{\partial^2 v}{\partial y^2}\right] \end{aligned}$$

及 Green 公式可得

$$\begin{aligned} &\int_C \left[u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\left(\alpha\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] dy \\ &- \left[u\left(\frac{\partial v}{\partial x} + \lambda\frac{\partial u}{\partial y}\right) + v\left(-\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y}\right)\right] dx \\ =&\iint\limits_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \lambda\left(\frac{\partial u}{\partial y}\right)^2 + \alpha\left(\frac{\partial v}{\partial x}\right)^2 + \mu\left(\frac{\partial v}{\partial y}\right)^2\right] dxdy \end{aligned}$$

$$+\iint_D \left[u\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 v}{\partial x\partial y}+\lambda\frac{\partial^2 u}{\partial y^2}\right)+v\left(\alpha\frac{\partial^2 v}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial y^2}\right)\right]dxdy,$$

由 (I)′ 及边界条件可知

$$\iint_D \left[\left(\frac{\partial u}{\partial x}\right)^2+\lambda\left(\frac{\partial u}{\partial y}\right)^2+\alpha\left(\frac{\partial v}{\partial x}\right)^2+\mu\left(\frac{\partial v}{\partial y}\right)^2\right]dxdy=0,$$

即得 $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$, 于是立得 $u=v=0$.

2) 标准型 (II) 所对应的方程组为

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 v}{\partial x\partial y}+\lambda\frac{\partial^2 u}{\partial y^2}=0,\\ \alpha\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial y^2}=0.\end{cases}\tag{II$'$}$$

从等式

$$\begin{aligned}&\frac{\partial}{\partial x}\left[u\left(\frac{\partial u}{\partial x}+s\frac{\partial v}{\partial y}\right)+v\left(\alpha\frac{\partial v}{\partial x}+t\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\Bigg[u\left((2-s)\frac{\partial v}{\partial x}+\lambda\frac{\partial u}{\partial y}\right)\\&+v\left((2-t)\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)\Bigg]\\=&\Bigg[\left(\frac{\partial u}{\partial x}\right)^2+(2+s-t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\mu\left(\frac{\partial v}{\partial y}\right)^2+\alpha\left(\frac{\partial v}{\partial x}\right)^2\\&+(2+t-s)\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\lambda\left(\frac{\partial u}{\partial y}\right)^2\Bigg]+u\left[\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 v}{\partial x\partial y}+\lambda\frac{\partial^2 u}{\partial y^2}\right]\\&+v\left[\alpha\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial y^2}\right]\end{aligned}$$

及 Green 公式, 由 (II)′ 及边界条件可知

$$\begin{aligned}&\iint_D \Bigg[\left(\frac{\partial u}{\partial x}\right)^2+(2+s-t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\mu\left(\frac{\partial v}{\partial y}\right)^2+\alpha\left(\frac{\partial v}{\partial x}\right)^2\\&+(2-s+t)\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\lambda\left(\frac{\partial u}{\partial y}\right)^2\Bigg]dxdy=0.\end{aligned}\tag{1.6}$$

取

$$t-s=2\frac{(\alpha\lambda)^{\frac{1}{2}}-\mu^{\frac{1}{2}}}{(\alpha\lambda)^{\frac{1}{2}}+\mu^{\frac{1}{2}}},$$

由于

$$(2+s-t)^2=\frac{16\mu}{\alpha\lambda+\mu+2(\alpha\lambda\mu)^{\frac{1}{2}}}<\frac{16\mu}{4}=4\mu,$$

$$(2-s+t)^2=\frac{16\alpha\lambda}{\alpha\lambda+\mu+2(\alpha\lambda\mu)^{\frac{1}{2}}}<\frac{16\alpha\lambda}{4}=4\alpha\lambda,$$

所以 (1.6) 的积分号下是一定正的二次型. 因此得出 $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$, 即 $u=v=0$. 明所欲证.

附记 1　对于 m 个自变数, n 个未知函数的情况, 我们可得到在 §1 提及的类似的必要条件和 Green 公式; 在某些附加条件下, 由 Green 公式可以证明 Dirichlet 问题的解的唯一性.

附记 2　А. В. Бицадзе举出的例子 (见 [3], p.361)

$$A=\begin{pmatrix}1&0\\2&1\end{pmatrix},\quad B=0,\quad C=\begin{pmatrix}-1&-2\\0&-1\end{pmatrix}.$$

当 $P=I$ 时虽然不适合 (A), 但取 $P=\begin{pmatrix}-1&2\\-2&1\end{pmatrix}$, 则它依然适合于条件 (A). 若左乘以 $\begin{pmatrix}-2&2\\-3&2\end{pmatrix}$, 右乘 $\begin{pmatrix}1&-1\\-\frac{1}{2}&1\end{pmatrix}$, 则它实质上与

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0,\\[2mm]\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 v}{\partial y^2}=0\end{cases}$$

等价. 这方程组的解的唯一性是显然的. 由此可见标准型的重要性.

附记 3　四个标准型各有其代数特性, 可以研究它们是否还各有解析特性.

附记 4　非强椭圆型的椭圆型微分方程组的唯一性定理是不对的, 但在 §1 中我们仅举出对十分特殊的区域不对. 是否对一般区域都不对呢, 因此可以提出以下的问题: 哪些区域唯一性定理对, 哪些不对? 如果解不唯一, 试求出所有的解来.

参考文献

[1] Вищик М. И., О сильно эллиптических системах дифференциальных уравнений. Мат. сб., 1951, 29: 613-676.

[2] 丁夏畦, 王康廷, 马汝念, 张同, 孙家乐. 常系数二阶椭圆型微分方程组的定义. 数学学报, 1960, 10: 276-287.

[3] Бицадзе А. В. 线性偏微分方程的某些线性定解问题. 数学进展, 1958, 4: 321-403.

Лаврентьев的混合型方程*

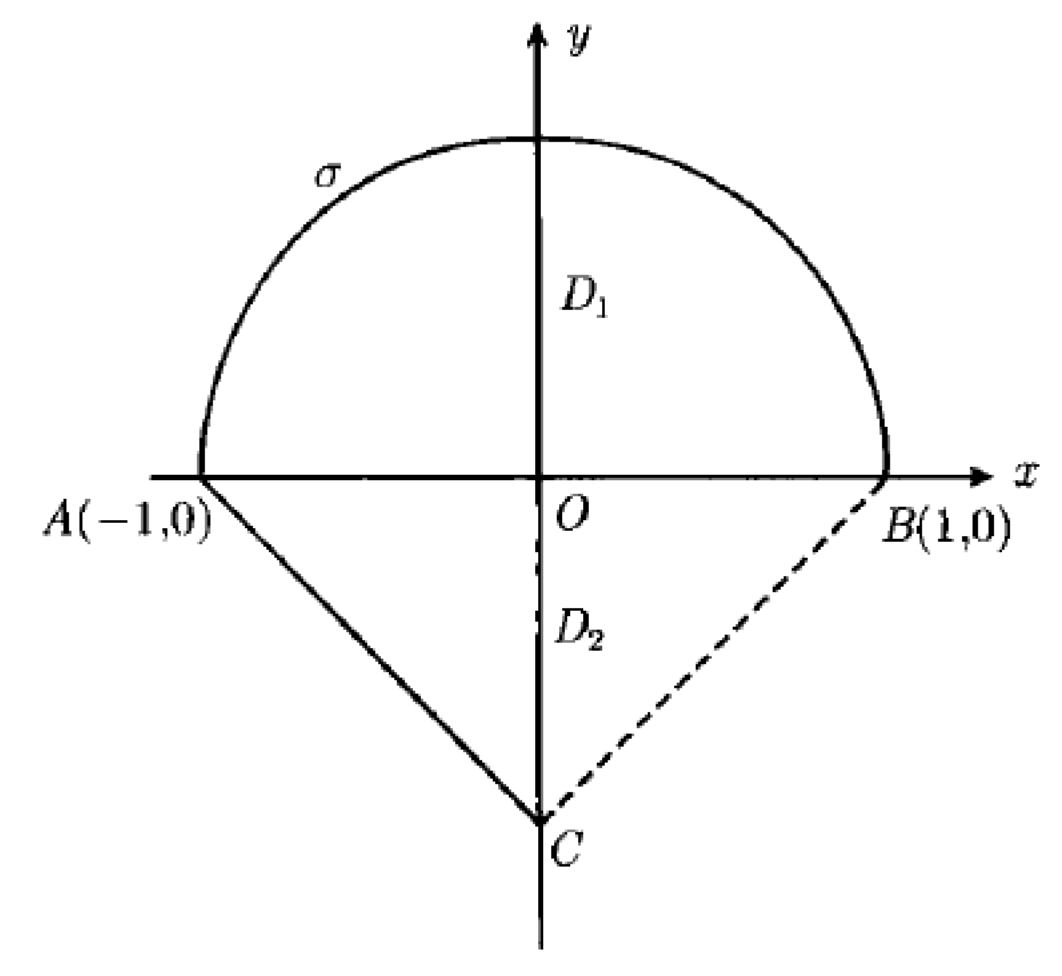
§1. 引　　言

在研究空间混合型微分方程的过程中, 同时也就改进了平面混合型偏微分方程的工作. 现在以M. A. Лаврентьев的方程为例说明之如次：我们只讲些不太失去普遍性的特例, 把推到更复杂的研究留给读者. Лаврентьев方程是

$$\frac{\partial^2 U}{\partial x^2}+\mathrm{sgn}y\frac{\partial^2 U}{\partial y^2}=0, \tag{1.1}$$

这儿 $\mathrm{sgn}y=1$, 当 $y>0$; $\mathrm{sgn}y=-1$, 当 $y<0$.

所研究的问题是 Tricomi 问题. D 是 x,y 平面上的单联域. 它在上半平面 $y>0$ 的边界是以 $A(-1,0), B(1,0)$ 为直径的圆弧 σ, 在下半平面 $y<0$, D 的边界是方程 (1.1) 的两条特征线: $x+y=-1, x-y=1$.



D_1, D_2 分别表示 D 在 $y>0$ 及 $y<0$ 上的部分. AB 直线称为变型线.

A. B. Бицадзе[1] 解决了以下的 Tricomi 问题. 求出适合于以下条件的函数 U:

(i) 在 D 内当 $y\neq 0$ 时, U 适合于 (1.1);

(ii) 在闭域 $\bar{D}$ 上 U 是连续函数;

(iii) 偏微商 $\frac{\partial U}{\partial x}$ 与 $\frac{\partial U}{\partial y}$ 在 D 的内部连续, 而且在 A,B 点附近可以有低于一级的无穷大;

(iv) 在曲线 σ 上及一条特征线上, 例如 AC 上, U 取已知值

$$U=\varphi,\qquad 当\ (x,y)\ 在\ \sigma\ 上, \tag{1.2}$$

$$U=\psi(x),\quad 当\ (x,y)\ 在\ AC\ 上. \tag{1.3}$$

当然假定在 A 点这二函数的数值相吻合, 同时假定这两个函数 φ,ψ 适合 Hölder 条件.

---

* 1964 年 5 月 20 日收到, 1964 年 10 月 28 日收到修改稿. 发表于《数学学报》, 1965, 15(6): 873-882.

本文的目的在于证明：1) 如果假定了 (ii), 则 (iii) 中关于无穷大阶的假定可以完全不要; 2) 即使放松了条件 (ii), (iii) 中关于无穷大阶的假定还是可以改善, 本文已经获得了最优的结果.

为了使读者易于接受, 我们一切从头讲起.

§ 2. 关于复变函数的引理

定理 1　一个在全平面上除 $z=0, z=\infty$ 外处处解析而且在实轴虚轴上取实值的单值函数一定有以下的 Laurent 展开式

$$g(z)=\sum_{n=-\infty}^{\infty} b_{2n}z^{2n}, \tag{2.1}$$

这儿 b_{2n} 是实数.

证　假定这函数的 Laurent 展式是

$$g(z)=\sum_{n=-\infty}^{\infty} a_n z^n, \tag{2.2}$$

由于在实轴上 $\mathscr{I}g(z)=0$, 由 Schwarz 对称原理立刻得出

$$g(\bar{z})=\overline{g(z)}. \tag{2.3}$$

由此得出

$$\begin{aligned}a_n&=\frac{1}{2\pi i}\int_{|z|=1} g(z)z^{-(n+1)}dz=\frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})e^{-in\theta}d\theta\\&=\frac{1}{2\pi}\int_0^{\pi}[g(e^{i\theta})e^{-in\theta}+g(e^{-i\theta})e^{in\theta}]d\theta\\&=\frac{1}{\pi}\int_0^{\pi}\mathscr{R}(g(e^{i\theta})e^{-in\theta})d\theta,\end{aligned}$$

由此 a_n 是实数.

再命 $z=iw$,

$$g(iw)=\sum_{n=-\infty}^{\infty} a_n i^n w^n,$$

由假定可知 $i^n a_n$ 也是实数, 因而得到本定理.

这定理不难推广为

定理 2　命 k 是一正整数. 适合以下条件的单值函数 $g(z)$:

(i) 在全平面上除 $z=0, z=\infty$ 外处处解析;

(ii) 在直线 $y = x\text{tg}\dfrac{l\pi}{k}, l = 0, 1, 2, \cdots, k-1$ 上取实值, 一定有以下的 Laurent 展式

$$g(z) = \sum_{n=-\infty}^{\infty} b_{kn} z^{kn}.$$

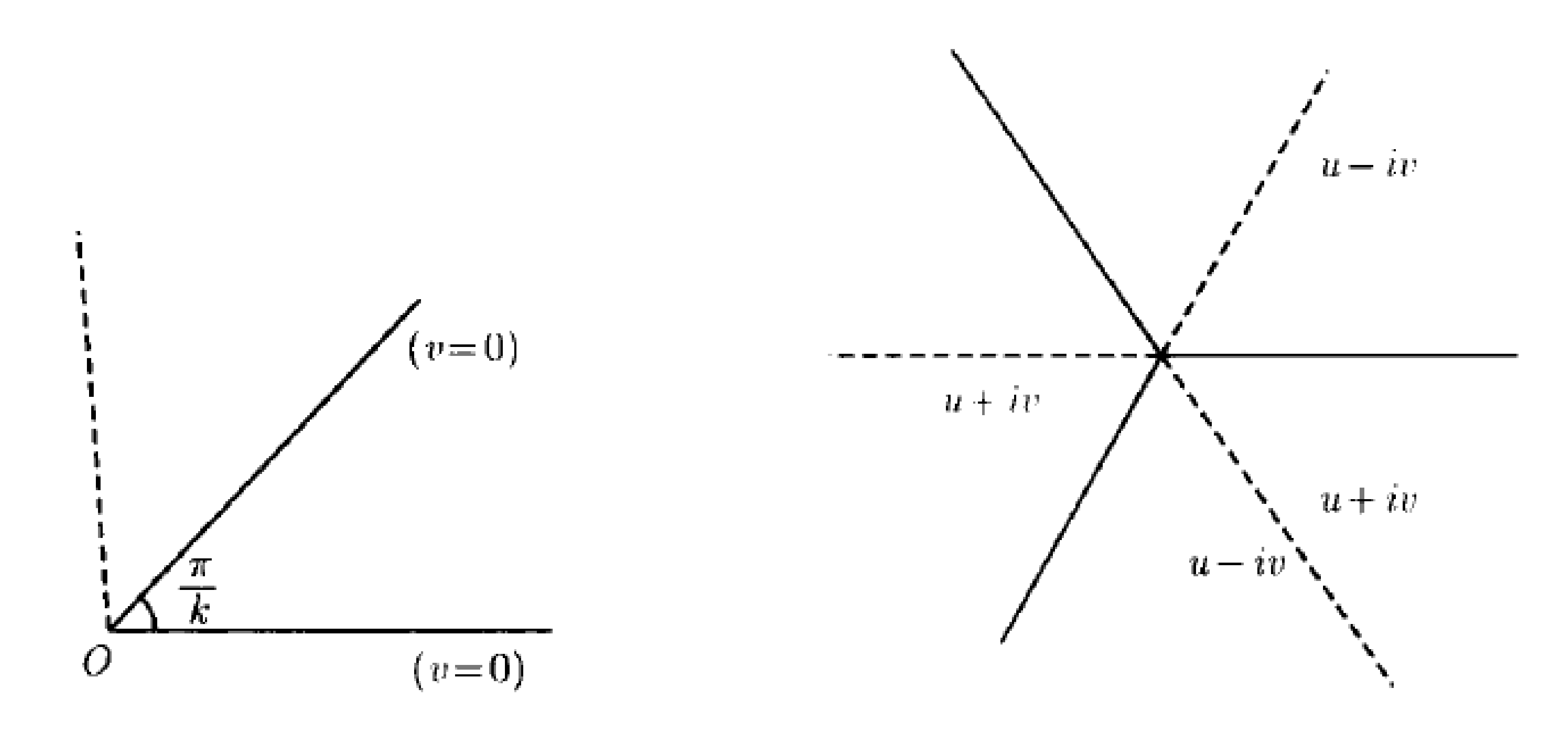
这个定理的证明是显然平行于定理 1.

但条件 (ii) 可以减弱为

(ii′) 在两条射线 $y = 0, 0 \leqslant x < \infty$; 及 $y = x\text{tg}\dfrac{\pi}{k}, 0 < x < \infty$ 上函数 $g(z)$ 取实值即足.

依 Schwarz 的对称原理, 依 $y = x\text{tg}\dfrac{\pi}{k}$ 反射, 因此在 $y = x\text{tg}\dfrac{2\pi}{k}$ 上 $v = 0$. 继续行使推出 (ii) 来.

注意: 刚才反射的角度是 $\dfrac{2\pi}{2k}$. 如果反射的角度是 2π 的奇数分之一, 便如何? 如图所示, 经角度 $\dfrac{2\pi}{3}$ 连续反射三次后便与 $\dfrac{2\pi}{6}$ 的反射相同了. 因此, $\dfrac{2\pi}{k}$ 的情况与 $\dfrac{2\pi}{2k}$ 的情况相同, 更一般些, 凡是以 2π 的有理倍数都可以如此处理.



附记 如果两条线的夹角与 2π 不可通度量, 在这两条射线上都取实值. 则反射再反射得出无穷条射线, 其上函数 $g(z)$ 都取实值. 这些值线与 x 轴的交角在 $(0, 2\pi)$ 中处处稠密. 如果 $g(z)$ 是单值函数, 由连续性可知它是一实值函数, 唯一可能是 $g(z)$ 是一实常数.

§ 3. 边界值等于零时的通解

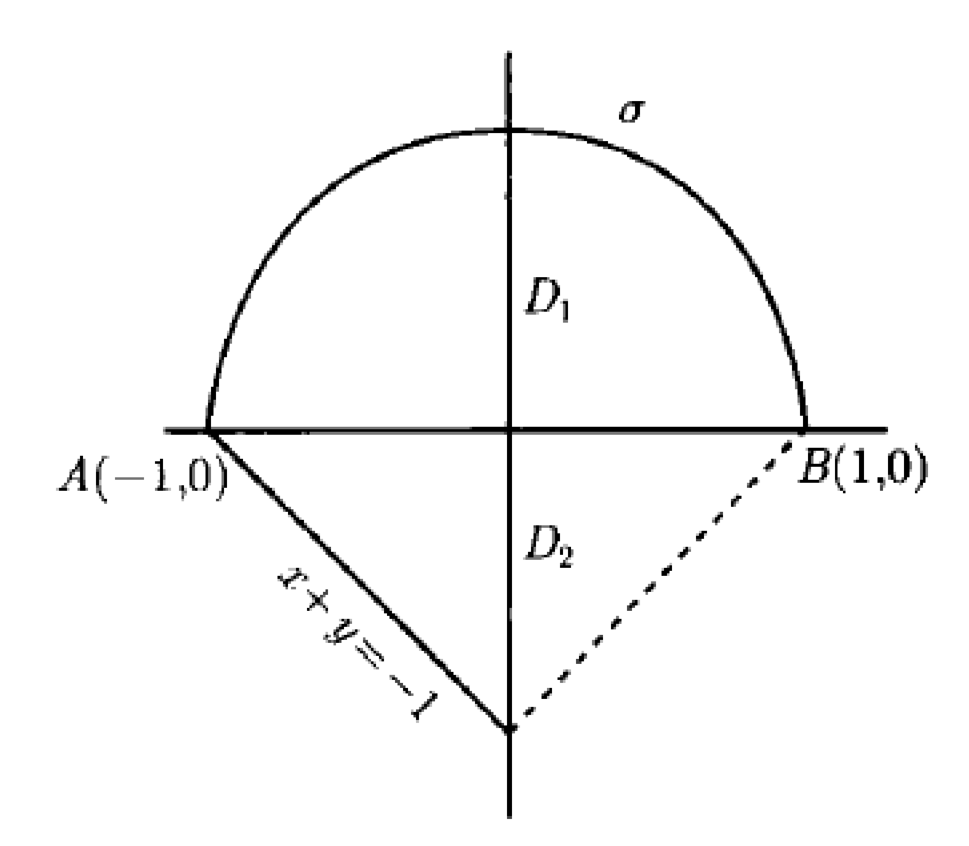
我们提出条件远较Бицадзе为少的问题:

问题 I 求适合于以下条件的函数 U:

(i) 在 D 内, 当 $y \neq 0$ 时, U 适合于 (1.1);

(ii) 除 $z=\pm 1$ 外, U 是闭域 $\overline{D}$ 上的连续函数;

(iii) $\dfrac{\partial U}{\partial x}, \dfrac{\partial U}{\partial y}$ 在 D 内部连续;



(iv) 在上半圆周上 $U=0$, 即

$$U\bigg|_{\substack{x^2+y^2=1\\ y>0}}=0. \tag{3.1}$$

在线段 $x+y=-1$ 上 $U=0$, 即

$$U\bigg|_{\substack{x+y=-1\\ -1<y<0}}=0. \tag{3.2}$$

解　在 D_2 上

$$U(x,y)=f(x+y)+g(x-y),\quad f(-1)=0, \tag{3.3}$$

此处 $f(t)$ 与 $g(t)$ 是在 $[-1,1]$ 上的任意两个连续函数, 而且在 $(-1,1)$ 上有二阶微商. 由条件 (3.2) 可知

$$g(2x+1)=0,\quad -1\leqslant x\leqslant 0,$$

即得 $g(\xi)=0$, 当 $-1\leqslant\xi\leqslant 1$. 因此在 D_2 上

$$U(x,y)=f(x+y),\quad (x,y)\in D_2.$$

由此推出

$$\frac{\partial U}{\partial x}-\frac{\partial U}{\partial y}=0. \tag{3.4}$$

由假定 (iii) 这条件在 AB 上 (A,B 二点除外) 也成立.

在 D_1 中, U 是调和函数. 命 $V(x,y)$ 是在 D_1 中与 $U(x,y)$ 共轭的函数 (可能差一常数待定), 由 Cauchy-Riemann 条件及 (iii), 由 (3.4) 得出

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial x}=0\quad (y=0,|x|<1).$$

积分之可得

$$U(x,0)+V(x,0)=C\quad(|x|<1),$$

取合适的 V 使 $C=0$.

因此问题一变而为：求调和函数 $U(x,y)$, 适合于

$$\left.\begin{array}{l}U|_\sigma=0,\\ U(x,0)+V(x,0)=0\quad(|x|<1).\end{array}\right\}\tag{3.5}$$

研究复变函数

$$W=f(z)=U+iV$$

这定义一保角变换, 把 x 轴变为直线 $U+V=0$, 把上半圆弧 σ 变为虚轴 $U=0$. 对 x 轴行 Schwarz 对称原理, 得

$$f(\bar{z})=-i\overline{W};$$

对上半圆弧 σ 行 Schwarz 对称原理, 得

$$f\left(\frac{1}{\bar{z}}\right)=-U+iV=-\overline{W}.$$

从上半圆内任一点出发, 对上半圆, 对圆外实轴, 对下半圆, 对圆内实轴行四次对称, 回到了原来的出发点. 其函数值的变化是

$$W\to-\overline{W}\to-(\overline{-i\overline{W}})=-iW\to i\overline{W}\to i(\overline{-i\overline{W}})=-W.$$

即当 z 绕 -1(或 $+1$) 一周时, 函数值变号.

函数 $\sqrt{(1+z)/(1-z)}$ 也有此性质, 因此, 函数

$$h(z)=(1+i)\sqrt{\frac{1+z}{1-z}}f(z)$$

是一个单值函数, 仅在 $z=\pm1$ 二点可能是孤立奇点.

在圆内实轴上,

$$\mathscr{I}h(z)=0.$$

由于

$$\frac{1+e^{i\theta}}{1-e^{i\theta}}=\frac{\cos\dfrac{\theta}{2}}{-i\sin\dfrac{\theta}{2}}.$$

因此在上半个圆上

$$h(e^{i\theta}) = 2^{-\frac{1}{2}}(1+i)^2(U+iV)\sqrt{\operatorname{ctg}\frac{\theta}{2}}, \quad 0<\theta<\pi,$$

即得

$$\mathscr{T} h(e^{i\theta}) = \sqrt{2\operatorname{ctg}\frac{\theta}{2}}U\Big|_{\sigma} = 0.$$

也就是 $h(z)$ 是一个在上半个圆的边界上取实值的函数. 依 Schwarz 对称原理它在下半圆与圆外实轴上也取实值. 变形

$$W = \frac{1+z}{1-z},$$

把实轴变为实轴, 单位圆变为虚轴, 因此

$$g(W) = h(z)$$

是一个在两坐标轴上都取实值的单值函数, 因此由定理 2.1 可知

$$h(z) = \sum_{n=-\infty}^{\infty} b_{2n}\left(\frac{1+z}{1-z}\right)^{2n}, \quad b_{2n}\text{是实数}.$$

$$f(z) = U+iV = \frac{1-i}{2}\sum_{n=-\infty}^{\infty} b_{2n}\left(\frac{1+z}{1-z}\right)^{2n-\frac{1}{2}}, \tag{3.6}$$

即得

$$U = \frac{1}{2}\sum_{n=-\infty}^{\infty} b_{2n}\left(\mathscr{R}\left(\frac{1+z}{1-z}\right)^{2n-\frac{1}{2}} + \mathscr{T}\left(\frac{1+z}{1-z}\right)^{2n-\frac{1}{2}}\right). \tag{3.7}$$

这是问题 I 的一般解. 我们并未假定函数 U 在 $z=\pm 1$ 时的性质.

§ 4. 唯 一 性

一些偏微分方程工作者喜爱“唯一性”定理, 它等价于在什么条件下, (3.8) 恒等于“0”的问题.

我们现在提两个办法.

第一种办法假定 U 在 $\bar{D}$ 上连续. 也就是

$$f(z) = \frac{1-i}{2}\sum_{n=-\infty}^{\infty} b_{2n}\left(\frac{1+z}{1-z}\right)^{2n-\frac{1}{2}}$$

的实部在上半闭圆内连续. 命 $t=\dfrac{1+z}{1-z}$, 则函数

$$\frac{1-i}{2}\sum_{n=-\infty}^{\infty} b_{2n}t^{2n-\frac{1}{2}}$$

的实部在以下的区域内连续:

$$t=\rho e^{i\theta},\quad 0\leqslant\theta\leqslant\frac{\pi}{2},$$

再命 $W=t^{\frac{1}{2}}$, 则

$$g(W)=\frac{1-i}{2}\sum_{n=-\infty}^{\infty} b_{2n}W^{4n-1}$$

的实部在

$$W=\rho e^{i\theta},\quad 0\leqslant\theta\leqslant\frac{\pi}{4},\ \rho\geqslant 0$$

内连续.

命 $g(W)=p(W)+iq(W)$. 当 $\theta=\dfrac{\pi}{4}$ 时,

$$g(\rho e^{\pi i/4})=\frac{1-i}{2}e^{-\pi i/4}\sum_{n=-\infty}^{\infty}(-1)^n b_{2n}\rho^{2n-1}=-\frac{i}{\sqrt{2}}\sum_{n=-\infty}^{\infty}(\alpha)^n b_{2n}\rho^{4n-1},$$

即 $p(\rho e^{i\pi/4})=0$, 由连续性可知

$$\lim_{\rho\to 0} p(\rho e^{i\theta})=0,\quad \text{当 } 0\leqslant\theta\leqslant\frac{1}{4}\pi.$$

当 $\theta=0$ 时, 显然有 $p+q=0$. 利用 Schwarz 对称原理可见

$$\left.\begin{aligned}
&\lim_{\rho\to 0} p(\rho e^{i\theta})=0,\quad \text{当 } 0\leqslant\theta\leqslant\frac{1}{2}\pi,\\
&\lim_{\rho\to 0} q(\rho e^{i\theta})=0,\quad \text{当 } \frac{1}{2}\pi\leqslant\theta\leqslant\pi,\\
&\lim_{\rho\to 0} p(\rho e^{i\theta})=0,\quad \text{当 } \pi\leqslant\theta\leqslant\frac{3}{2}\pi,\\
&\lim_{\rho\to 0} q(\rho e^{i\theta})=0,\quad \text{当 } \frac{3}{2}\pi\leqslant\theta\leqslant 2\pi.
\end{aligned}\right\}\tag{4.1}$$

如果 $g(W)$ 有无穷个负幂项, 则 $W=0$ 是一本质奇点. 由 Weierstrass 定理, 在此点附近可以无限逼近 $1+i$, 这与 (4.1) 相矛盾. 同样处理有无穷个正幂项的情况. 如果只有有限个正负项, 则 $W=0$ 或 $W=\infty$ 是极点, 也是不可能的. 因此 $g(W)=0$. 因而得出“唯一性”.

这说明了: 如果我们保留了Бицадзе的条件 (ii), 则他所用的 0 条件可以完全不要.

第二种办法是在 $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$ 上加 0 条件. 切实些说, 如果假定了

$$\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} = o(|1+z|^{-\frac{3}{2}}) \quad (z \to -1) \tag{4.2}$$

及

$$\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} = o(|1-z|^{-\frac{5}{2}}) \quad (z \to 1) \tag{4.3}$$

便能保证唯一性. 这结果比Бицадзе的优异, 在我们讲授高等数学的过程中曾经改进了Келдыщ-Седов的结果, 因而已经推出比Бицадзе较好的结果, 但本文的结果又更进一步 (见《高等数学引论》第二卷讲稿, 中国科学技术大学讲义, 未正式出版).

由条件 (4.2) 可知

$$\frac{\partial U}{\partial x}, \frac{\partial V}{\partial x} = o(|1+z|^{-3/2}), \quad 1+z = \rho e^{i\theta}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

因此

$$f'(z) = \frac{d}{dx}(U+iV) = o(|1+z|^{-3/2})$$

与前相仿的对称原理可以推得: 对 $0 \leqslant \theta \leqslant 2\pi$, 常有

$$f'(z) = o(|1+z|^{-3/2}) \tag{4.4}$$

与

$$f'(z) = o(|1-z|^{-5/2}). \tag{4.5}$$

由 (3.6) 得

$$f'(z) = \frac{1-i}{2} \sum_{n=-\infty}^{\infty} b_{2n} \left(\frac{1+z}{1-z}\right)^{2n-\frac{3}{2}} (4n-1) \frac{1}{(1-z)^2}. \tag{4.6}$$

由 (4.4) 可知, 当 $2n - \frac{3}{2} \leqslant -\frac{3}{2}$(即 $n \leqslant 0$) 时

$$b_{2n} = 0,$$

又由 (4.5) 可知, 当 $-\left(2n - \frac{3}{2}\right) - 2 \leqslant -\frac{5}{2}$(即 $n \geqslant 1$) 时

$$b_{2n} = 0.$$

两者合计得, 由 (3.7) 可知

$$U = 0.$$

即得唯一性. 条件 (4.2),(4.3) 显然优于Бицадзе的条件.

这说明了即使放松了Бицадзе的条件 (ii), 他的 0 条件还能改进.

本方法的优点在于不仅改进了Бицадзе的 0 条件, 而在于找到了最佳的 0 条件. 不仅在于找到了最佳的 0 条件, 而在于找到了最一般的结果. 从而我们看出如果条件放宽到如何尺度, 便会出现怎样的解来.

§ 5. 存 在 性

仍如 §3, 但用较一般的 (1.2) 与 (1.3) 可以得出

$$g(2x+1)=\psi(x),\quad -1\leqslant x\leqslant 0,$$

即得

$$g(\xi)=\psi\left(\frac{\xi-1}{2}\right),\quad -1\leqslant \xi\leqslant 1.$$

即

$$U(x,y)=f(x+y)+\psi\left(\frac{x-y-1}{2}\right).$$

由此推出

$$\frac{\partial U}{\partial x}-\frac{\partial U}{\partial y}=\psi'\left(\frac{x-y-1}{2}\right).$$

在 $y=0$ 上

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial x}=\psi'\left(\frac{x-1}{2}\right),$$

积分得

$$U+V=2\psi\left(\frac{x-1}{2}\right)$$

(积分常数随 V 取定).

因此问题一变而为求适合于条件

$$U|_{\sigma}=\varphi(\theta),\quad 0<\theta<\pi, \tag{5.1}$$

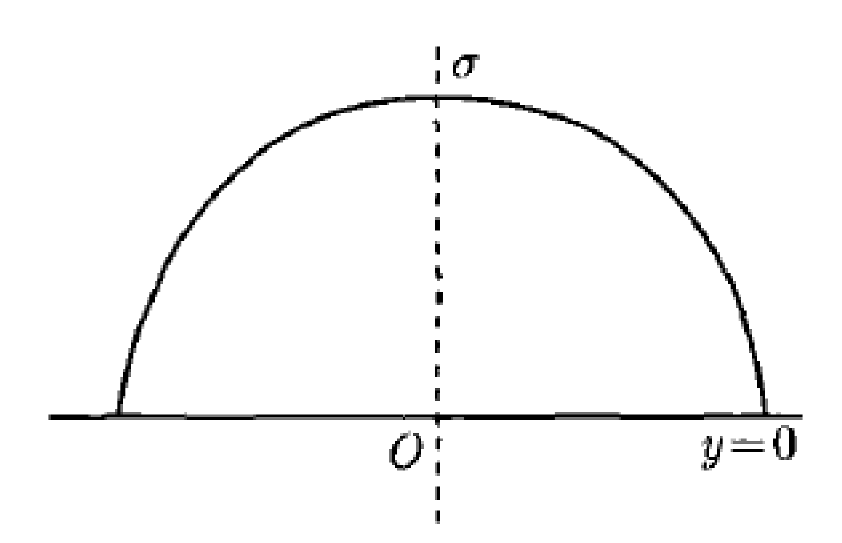
$$U+V|_{y=0}=2\psi\left(\frac{x-1}{2}\right) \tag{5.2}$$

的复函数

$$f(z)=U+iV.$$

考虑函数

$$h(z)=\sqrt{\frac{1+z}{1-z}}(1+i)f(z),$$



命 $v(z)=\mathscr{T}h(z)$, 则

$$v(z)|_{y=0}=\sqrt{\frac{1+x}{1-x}}(U+V)|_{y=0}=2\sqrt{\frac{1+x}{1-x}}\psi\left(\frac{x-1}{2}\right).$$

而

$$v(z)|_{\sigma}=\sqrt{2\operatorname{ctg}\frac{\theta}{2}}\varphi(\theta).$$

这是一个普通的 Dirichlet 问题. 但注意当 $\theta=0$ 时, 边界函数可能有奇性, 如此而已.

我们还是叙述一下其具体解法. 把问题一分为二.

$$v_1(z)|_{y=0}=0,\quad v_1(z)|_{\sigma}=\sqrt{2\operatorname{ctg}\frac{\theta}{2}}\varphi(\theta) \tag{5.3}$$

及

$$v_2(z)|_{y=0}=2\sqrt{\frac{1+x}{1-x}}\psi\left(\frac{x-1}{2}\right),\quad v_2(z)|_{\sigma}=0. \tag{5.4}$$

$v_1(z),v_2(z)$ 各为解析函数 $h_1(z)$ 与 $h_2(z)$ 的虚部. 显而易见 $v=v_1+v_2, h=h_1+h_2$. 由于 $\mathscr{R}h(z)|_{z=-1}=0$, 故可假定 $\mathscr{R}h_1(z)|_{z=-1}=\mathscr{R}h(z)|_{z=-1}=0$.

对于 $h_1(z)$, 由 Schwarz 对称原理可知

$$v_1(e^{-i\theta})=-\sqrt{2\operatorname{ctg}\frac{\theta}{2}}\varphi(\theta),\quad 0<\theta<\pi.$$

利用 Schwarz 公式, $h_1(z)$ 可由其虚部的边界值 $v_1(e^{i\theta})$ 表出:

$$\begin{aligned}h_1(z)&=\frac{i}{2\pi}\left(\int_0^{\pi}\sqrt{2\operatorname{ctg}\frac{\theta}{2}}\varphi(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta-\int_{-\pi}^{0}\sqrt{2\operatorname{ctg}\frac{-\theta}{2}}\varphi(-\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right)-C\\&=\frac{i}{2\pi}\int_0^{\pi}\sqrt{2\operatorname{ctg}\frac{\theta}{2}}\varphi(\theta)\frac{2z(e^{-i\theta}-e^{i\theta})}{1-2z\cos\theta+z^2}d\theta-C\\&=\frac{2z}{\pi}\int_0^{\pi}\sqrt{2\operatorname{ctg}\frac{\theta}{2}}\varphi(\theta)\frac{\sin\theta}{1-2z\cos\theta+z^2}d\theta-C,\end{aligned}$$

其中 C 为实常数. 由于 $\mathscr{R}h_1(z)|_{z=-1}=0$, 所以

$$C=\frac{-2}{\pi}\int_0^{\pi}\sqrt{2\mathrm{ctg}\frac{\theta}{2}}\frac{\sin\theta}{2(1+\cos\theta)}d\theta=\frac{-\sqrt{2}}{\pi}\int_0^{\pi}\sqrt{\mathrm{tg}\frac{\theta}{2}}\varphi(\theta)d\theta.$$

因此, 合并之得

$$\begin{aligned}h_1(z)&=\frac{2}{\pi}\int_0^{\pi}\sqrt{2\mathrm{ctg}\frac{\theta}{2}}\varphi(\theta)\frac{\sin\theta(1+z)^2}{2(1+\cos\theta)(1-2z\cos\theta+z^2)}d\theta\\&=\frac{(1+z)^2}{\pi}\int_0^{\pi}\sqrt{2\mathrm{tg}\frac{\theta}{2}}\varphi(\theta)\frac{d\theta}{1-2z\cos\theta+z^2}.\end{aligned}\tag{5.5}$$

不难证明当 $\varphi(\theta)$ 满足 Hölder 条件时 $\mathscr{T}h_1(z)$ 满足 (5.3), 且 $\frac{1}{1+i}\sqrt{\frac{1-z}{1+z}}h_1(z)$ 适合 §1 中的条件 (iii).

再对 $h_2(z)$ 用 Schwarz 对称原理, 可知

$$v_2(z)|_{y=0}=-2\sqrt{\frac{1+\frac{1}{x}}{1-\frac{1}{x}}}\psi\left(\frac{\frac{1}{x}-1}{2}\right),\quad |x|>1.$$

这是上半平面的 Dirichlet 问题.

利用 Schwarz 公式有

$$\begin{aligned}h_2(z)&=\frac{1}{\pi}\int_{-\infty}^{\infty}v_2(t)\frac{1}{t-z}dt-C\\&=\frac{1}{\pi}\int_{-1}^{1}2\sqrt{\frac{1+t}{1-t}}\psi\left(\frac{t-1}{2}\right)\frac{dt}{t-z}\\&\quad-\frac{1}{\pi}\left(\int_{-\infty}^{-1}+\int_1^{\infty}\right)2\sqrt{\frac{1+\frac{1}{t}}{1-\frac{1}{t}}}\psi\left(\frac{\frac{1}{t}-1}{2}\right)\frac{dt}{t-z}-C\\&=\frac{1}{\pi}\int_{-1}^{1}2\sqrt{\frac{1+t}{1-t}}\psi\left(\frac{t-1}{2}\right)\frac{dt}{t-z}-\frac{1}{\pi}\int_{-1}^{1}2\sqrt{\frac{1+\tau}{1-\tau}}\psi\left(\frac{\tau-1}{2}\right)\frac{d\tau}{\tau-\tau^2z}-C\\&=\frac{2z}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\psi\left(\frac{t-1}{2}\right)\frac{1-t^2}{t(t-z)(1-tz)}dt-C,\end{aligned}$$

其中 C 是实常数, 由于 $\mathscr{R}h_2(z)|_{z=-1}=0$, 所以

$$C=\frac{-2}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\psi\left(\frac{t-1}{2}\right)\frac{dt}{t},$$

此处积分都取 Cauchy 主值.

合并之得

$$h_2(z) = \frac{2(1+z)^2}{\pi} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \psi\left(\frac{t-1}{2}\right) \frac{dt}{(t-z)(1-tz)}. \tag{5.6}$$

同样可以证明当 ψ 满足 Hölder 条件时 $\mathscr{T} h_2(z)$ 适合 (5.4), 且 $\dfrac{1}{1+i}\sqrt{\dfrac{1-z}{1+z}} h_2(z)$ 适合 §1 中的条件 (iii).

以上求出了 D_1 中的 $U(x,y)$. 而在 D_2 中它将由

$$U(x,y) = U(x+y,0) - \psi\left(\frac{x+y-1}{2}\right) + \psi\left(\frac{x-y-1}{2}\right) \tag{5.7}$$

给出.

我们假定了 $\varphi(\theta)$ 与 $\psi(t)$ 适合于 Hölder 条件, 于是由 (5.5) 与 (5.6) 所定义出的函数, 远较我们的唯一性所要求的函数类为狭仄. 因而提出 Hölder 条件能不能减弱的问题. 如果利用作者关于广义函数的概念 [2], 甚且可以看到, 把 φ(及 ψ) 作为某一类型的广义函数似乎也无不可. 看看可能性:

$$\left(\frac{1+z}{1-z}\right)^{-\frac{1}{2}} = (1-z)(1-z^2)^{-\frac{1}{2}} = (1-z)\sum_{k=0}^{\infty} \frac{\Gamma\left(k+\dfrac{1}{2}\right)}{k!\,\Gamma\left(\dfrac{1}{2}\right)} z^{2k}.$$

这函数在单位圆上有形式 Fourier 级数

$$\sum_{n=-\infty}^{\infty} C_n e^{in\theta} = (1-e^{i\theta})\sum_{k=0}^{\infty} \frac{\Gamma\left(k+\dfrac{1}{2}\right)}{k!\,\Gamma\left(\dfrac{1}{2}\right)} e^{2ik\theta},$$

这儿 $C_n \asymp n^{-\frac{1}{2}}$(当 $n \to \infty$). 因此, 如果超出了包括这函数的函数类是不可能有唯一性的. 但同时建议适合 $C_n = o(n^{-\frac{1}{2}})$ 的函数类大有可能. 既存在又唯一.

参 考 文 献

[1] 毕察捷 (A. B. Бицадзе). 混合型微分方程. 北京: 科学出版社, 1955.

[2] 华罗庚. 广义函数导引. 数学进展, 1963, 6: 391-409.

ON THE CLASSIFICATION OF THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

I. EQUIVALENCE

Let A, B, C denote three constant 2×2 matrices. Then

$$A\frac{\partial^2}{\partial x^2}\begin{pmatrix} u \\ v \end{pmatrix}+2B\frac{\partial^2}{\partial x\partial y}\begin{pmatrix} u \\ v \end{pmatrix}+C\frac{\partial^2}{\partial y^2}\begin{pmatrix} u \\ v \end{pmatrix}=0 \tag{I}$$

denote a system of two partial differential equations of the second order with two independent variables x, y and two unknown functions u,v. The system is denoted simply by(A, B, C).

Definition 1 (A, B, C) and (A_1, B_1, C_1) are said to be equivalent if one can be transformed into the other by means of successive applications of the following three kinds of operations: (i) $A_1 = p^2A+2pqB+q^2C$, $B_1 = prA+(ps+qr)B+qsC$, $C_1 = r^2A + 2rsB + s^2C$, $ps - qr \neq 0$(linear transformation of independent variables); (ii)$A_1 = PA$, $B_1 = PB$, $C_1 = PC$, $|P| \neq 0$(linear combination of equations), and (iii)$A_1 = AQ$, $B_1 = BQ$, $C_1 = CQ|Q| \neq 0$(linear transformation of unknown functions).

Evidently, to study a system of differential equations is equivalent to studying one of its equivalences.

Definition 2 A system is said to be reducible if it is equivalent to

$$A_1=\begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix},\quad B_1=\begin{pmatrix} b_1 & 0 \\ b_3 & b_4 \end{pmatrix},\quad C_1=\begin{pmatrix} c_1 & 0 \\ c_3 & c_4 \end{pmatrix}.$$

To study a reducible system is equivalent to studying two simple equations succesively. It is devoid of particular interest.

---

* Read at a symposium in celebration of the 5th anniversary of the University of Science and Technology of China held in 1963. Published in *Sci Sinica*(Notes), XIV(3): 461-465. authors: Hua Loo Keng(华罗庚)(*University of Science and Technology of China*), Wu Tze-chien(吴兹潜), Lin Wei(林伟)(*Sun Yet-sen University*)

Example (Бицадзе[1]) The system

$$A=\begin{pmatrix}1&0\\2&1\end{pmatrix},\quad B=0,\quad C=\begin{pmatrix}-1&-2\\0&-1\end{pmatrix}.$$

is reducible. In fact, multiplying the left by $\begin{pmatrix}-2&2\\-3&2\end{pmatrix}$ and the right by $\begin{pmatrix}1&-1\\-\frac{1}{2}&1\end{pmatrix}$, we have

$$A_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B_1=0,\quad C_1=\begin{pmatrix}1&0\\1&1\end{pmatrix},$$

It is not difficult to prove

Theorem 1 *A necessary and sufficient condition for a system to be reducible is that there are two non-zero vectors(a,b) and (c,d) such that*

$$(a,b)A=\alpha(c,d),\quad (a,b)B=\beta(c,d),\quad (a,b)C=r(c,d).$$

II. Classification of Biquadratic Forms

Definition 3 The determinant

$$Q(\xi,\eta)=|A\xi^2+2B\xi\eta+C\eta^2| \tag{II}$$

is called the biquadratic characteristic form of the system (A, B, C).

Evidently, (ii) and (iii) carry (II) into itself apart from a constant multiple; and by(i) we have

$$\begin{aligned}Q_1(\xi,\eta)&=|A_1\xi^2+2B_1\xi\eta+C_1\eta^2|\\&=|A(p\xi+r\eta)^2+2BC(p\xi+r\eta)(q\xi+s\eta)+C(q\xi+s\eta)^2|\\&=Q(\xi',\eta'),\end{aligned}$$

where

$$\xi'=p\xi+r\eta,\quad \eta'=q\xi+s\eta.$$

In order to classify (I), we shall classify Q(ξ,η) first. By means of elementary algebra, (II) is equivalent to the following two standard forms:

$$Q(\xi,\eta)=(\xi^2+\varepsilon\eta^2)(\xi^2+k^2\eta^2), \tag{III}$$

where

$$\varepsilon=1,\begin{cases}0<k<1 \text{ for } Q \text{ having two gdistinct pairs of complex roots,}\\ k=1 \text{ for } Q \text{ having a pair of double complex roots,}\\ k=0 \text{ for } Q \text{ having a pair of complex roots and a double real root;}\end{cases} \tag{2.1}$$

$$\varepsilon = 0,\ k = 0 \text{ for Q having a quadruple real root;} \tag{2.2}$$

and

$$Q(\xi,\eta) = \xi\eta(\delta\xi^2 + 2\alpha\xi\eta + \varepsilon\eta^2), \tag{IV}$$

where

$$\delta = \varepsilon = 1,\begin{cases}0 \leqslant \alpha < 1 \text{ for Q having a pair of complex and two distinct}\\ \qquad\qquad \text{real roots,}\\ \alpha = 1 \text{ for Q having three distinct real roots,}\\ \alpha > 1 \text{ for Q having four distinct real roots;}\end{cases} \tag{2.3}$$

$$\delta = 1, \varepsilon = \alpha = 0 \text{ for Q having a triple and a simple real root;} \tag{2.4}$$

$$\delta = \varepsilon = 0, \alpha = 1 \text{ for Q having two double real roots.} \tag{2.5}$$

III.Standard Form

Theorem 2 *An irreducible system with the characteristic form* (III) *is equivalent to*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad B = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix},\quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \tag{A}$$

where $\lambda + \mu - 4b = k^2 + \varepsilon,\quad \lambda\mu = k^2\varepsilon,\quad b \neq 0$ *(for* $\lambda = \mu$ *we may further assume that* $b<0$).

Theorem 3 *An irreducible system with the characteristic form* (IV) *is equivalent to*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad B = \begin{pmatrix} \frac{\varepsilon}{2c} & 1 \\ b & \frac{\delta}{2c} \end{pmatrix},\quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{B}$$

where

$$b = \frac{\delta\varepsilon}{4c^2} - \frac{\alpha}{2c} + \frac{1}{4} \neq 0,\quad c \neq 0.$$

Definition 4 (Петровский) The system with (II) possessing complex roots is called elliptic.

Including the reducible cases we have

Theorem 4 Петровский's *elliptic system has the following standard forms*:

$$A=\begin{pmatrix}1&0\\0&\alpha\end{pmatrix},\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad C=\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix},\tag{3.1}$$

$\alpha\lambda+\mu+4=\lambda\mu+\alpha,\quad \alpha>0,\quad \lambda\mu>0;$

$$A=\begin{pmatrix}1&0\\0&\alpha\end{pmatrix},\quad B=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad C=\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix},\tag{3.2}$$

$\alpha\lambda+\mu-4=\lambda\mu+\alpha,\quad \alpha>0,\quad \lambda\mu>0;$

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad C=\begin{pmatrix}1&0\\0&\mu\end{pmatrix},\quad \mu>0;\tag{3.3}$$

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad C=\begin{pmatrix}1&0\\0&\mu\end{pmatrix},\quad \mu>0.\tag{3.4}$$

IV. Strongly Elliptic Type

If there exist β and γ such that

$$|A+2\beta B+\gamma C|=0,\quad \text{for } \beta^2<\gamma,\tag{4.1}$$

then we have a non-zero vector $\begin{pmatrix}a\\b\end{pmatrix}$ such that

$$(A+2\beta B+\gamma C)\begin{pmatrix}a\\b\end{pmatrix}=0.$$

Consequently the system (I) has a solution

$$u=a(x^2+2\beta xy+\gamma y^2-1),\quad v=b(x^2+2\beta xy+\gamma y^2-1),$$

both of which vanish on the ellipse $x^2+2\beta xy+\gamma y^2=1$. It shows that Петровский's definition does not imply the uniqueness of the solution of the Dirichlet problem of the system of differential equations.

In 1960, Ting and others[1] gave a neat condition for the uniqueness of elliptic system: For $\beta^2<\gamma$,

$$|A+2\beta B+\gamma C|\neq 0.\tag{N}$$

Definition 5 A system satisfying (N) is said to be of the strongly elliptic type.

Theorem 5 *Evidently* (3.2),(3.3),(3.4) *are of the strongly elliptic type*;(3.1)*is of the strongly elliptic type if and only if* $\lambda > 0$.

Proof For(3.1), we have

$$|A+2\beta B+\gamma C| = \left|\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} + 2\beta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \gamma\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right|$$
$$= (1+\gamma\lambda)(\alpha+\gamma\mu)+4\beta^2.$$

If $\lambda < 0$, we take $\beta = 0, \gamma = -\dfrac{1}{\lambda}$such that $|A+2\beta B+\gamma C| = 0$. If $\lambda > 0$, it is always > 0.

The following more complicated condition for the system of strongly elliptic type is due to Вищик[2]:

There is a matrix P such that $A_1 = PA, B_1 = PB, C_1 = PC$ satisfy

$$\widetilde{A}_1 + 2\widetilde{B}_1 t + \widetilde{C}_1 t^2 > 0, \tag{M}$$

where $\widetilde{H} = \dfrac{1}{2}(H+H')$ and H' is the transposed matrix of H.

For (3.1) (with $\lambda > 0$), (3.2), (3.3), (3.4) we have evidently

$$\widetilde{A}_1 + 2\widetilde{B}_1 t + \widetilde{C}_1 t^2 > 0. \tag{N}$$

Therefore condition(M) and condition (N) have no significant difference.

Remark 1 A. B. Бицадзе[1] gave the following example

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$$

to show that condition (M) is sufficient but not necessary for the uniqueness of the Dirichlet problem. However, simply multiplying on the left by $\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$, we have a system satisfying (M).

Remark 2 Starting with the standard forms (A) and (B), two of the authors obtained some results concerning the other types of the systems which will appear elsewhere later.

Remark 3 The uniqueness of the solution of the Dirichlet problem of (3.1) with $\lambda > 0$ and (3.2) follows immediately from Green's formula and the following identities: for(3.1), we use

$$\frac{\partial}{\partial x}\left[u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+v\left(\alpha\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[u\left(\frac{\partial v}{\partial x}+\lambda\frac{\partial u}{\partial y}\right)+v\left(-\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)\right]$$
$$=\left(\frac{\partial u}{\partial x}\right)^2+\lambda\left(\frac{\partial v}{\partial y}\right)^2+\alpha\left(\frac{\partial v}{\partial x}\right)^2+\mu\left(\frac{\partial v}{\partial y}\right)^2$$
$$+u\left[\frac{\partial^2 u}{\partial x^2}+\tau\frac{\partial^2 v}{\partial x\partial y}+\lambda\frac{\partial^2 v}{\partial y^2}\right]+v\left[\alpha\frac{\partial^2 v}{\partial x^2}-\tau\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial y^2}\right],$$

and for(3.2), we use

$$\frac{\partial}{\partial x}\left[u\left(\frac{\partial u}{\partial x}+s\frac{\partial v}{\partial y}\right)+v\left(\alpha\frac{\partial v}{\partial x}+t\frac{\partial u}{\partial y}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[u\left((\tau-s)\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}\right)+v\left((\tau-t)\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)\right]$$
$$=\left[\left(\frac{\partial u}{\partial x}\right)^2+(\tau+s-t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\mu\left(\frac{\partial v}{\partial y}\right)^2+\alpha\left(\frac{\partial v}{\partial x}\right)^2\right.$$
$$\left.+(\tau+t-s)\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\lambda\left(\frac{\partial u}{\partial y}\right)^2\right]$$
$$+u\left[\frac{\partial^2 u}{\partial x^2}+\tau\frac{\partial^2 v}{\partial x\partial y}+\lambda\frac{\partial^2 u}{\partial y^2}\right]+v\left[\alpha\frac{\partial^2 v}{\partial x^2}+\tau\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial y^2}\right]$$
$$(t-s=\tau(\sqrt{\alpha\lambda}-\sqrt{\mu})/(\sqrt{\alpha\lambda}+\sqrt{\mu})).$$

References

[1] Дин Ся-си, МаЖу-нянь, Ван Кон-тин, и Чжон Тун. *Acta Math. Sinica*, 1960, 10: 276-287.

[2] Вищик М. И. Мат. сб., 1951, 29: 613-676.

《华罗庚文集》已出版书目

（按出版时间排序）

1 华罗庚文集·数论卷 I 王元 审校 2010 年 5 月
2 华罗庚文集·数论卷 II 贾朝华 审校 2010 年 5 月
3 华罗庚文集·数论卷 III 王元 潘承彪 贾朝华 编译 2010 年 5 月
4 华罗庚文集·代数卷 I 万哲先 审校 2010 年 5 月
5 华罗庚文集·多复变函数论卷 I 陆启铿 审校 2010 年 5 月
6 华罗庚文集·应用数学卷 I 杨德庄 主编 2010 年 5 月
7 华罗庚文集·应用数学卷 II 杨德庄 主编 2010 年 5 月
8 华罗庚文集·代数卷 II 李福安 审校 2011 年 1 月
9 华罗庚文集·多复变函数论卷 II 周向宇 审校 2013 年 1 月

(O-4949.0101)

中国科学院华罗庚数学重点实验室丛书

华罗庚文集｜多复变函数论卷 II｜

www.sciencep.com



科学出版中心 数理分社
电话：(010) 64033664
Email：math-phy@mail.sciencep.com
网址：www.math-phy.cn

销售分类建议：高等数学

定 价：98.00 元